RELATIVE CONVERGENCE ESTIMATES FOR THE SPECTRAL ASYMPTOTIC IN THE LARGE COUPLING LIMIT

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ABSTRACT. We prove optimal convergence estimates for eigenvalues and eigenvectors of a class of singular/stiff perturbed problems. Our profs are constructive in nature and use (elementary) techniques which are of current interest in computational Linear Algebra to obtain estimates even for eigenvalues which are in gaps of the essential spectrum. Further, we also identify a class of "regular" stiff perturbations with (provably) good asymptotic properties. The Arch Model from the theory of elasticity is presented as a prototype for this class of perturbations. We also show that we are able to study model problems which do not satisfy this regularity assumption by presenting a study of a Schroedinger operator with singular obstacle potential.

1. INTRODUCTION

In this paper we give sharp estimates for the asymptotic behavior of the spectral problem for the family of self-adjoint operators $\mathbf{H}_\kappa$ which are defined by positive definite quadratic forms

$$\mathfrak{h}_\kappa(u,v) = \mathfrak{h}_b(u,v) + \kappa^2 \mathfrak{h}_e(u,v), \qquad u,v \in \mathcal{Q}(\mathfrak{h}_b) \subset \mathcal{Q}(\mathfrak{h}_e). \tag{1.1}$$

Here we have used $\mathcal{Q}(\mathfrak{h}_b)$ and $\mathcal{Q}(\mathfrak{h}_e)$ to denote the domain of definition of $\mathfrak{h}_b$ and $\mathfrak{h}_e$ and we assume that $\kappa^2 \to \infty$. Qualitative results for families of self-adjoint operators like $\mathbf{H}_\kappa$ have a long tradition. We are particularly influenced by the results from [28, 35]. Here by qualitative results we mean those results which prove (e.g.) that the spectral projections $E_\kappa(\cdot)$, $\mathbf{H}_\kappa = \int \lambda \, dE_\kappa(\lambda)$ converge in some appropriate sense.

To give a first idea of what is hidden within the abstract formulation (1.1) let us consider two simple examples that are representative for more complex model problems (studied later on in Section 5). The family of quadratic forms

$$\mathfrak{h}_\kappa(u,v) = \int_0^2 u'v' \, dx + \kappa^2 \int_1^2 u'v' \, dx, \quad u,v \in H_0^1[0,2], \ \ \kappa \to \infty \tag{1.2}$$

is paradigmatic for a regularly perturbed family, whereas the family

$$\mathfrak{h}_\kappa(u,v) = \int_0^2 u'v' \, dx + \kappa^2 \int_1^2 uv \, dx, \quad u,v \in H_0^1[0,2], \ \ \kappa \to \infty \tag{1.3}$$

is representative for the quadratic forms which violate our new regularity assumption. Note that in our relative theory the unbounded perturbation $\mathfrak{h}_e$ in (1.2) is preferable to the bounded perturbation $\mathfrak{h}_e$ in (1.3). Here we have used $H_0^1(\cdot)$ to denote the standard Sobolev spaces.

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The limit of the families like (1.2) and (1.3) can be a non-densely defined operator and we use the theory of [35] to study the convergence of such $\mathfrak{h}_\kappa$ and associated $\mathbf{H}_\kappa$ as $\kappa \to \infty$. Let now the operator $\mathbf{H}_\infty$ (in general non-densely defined) be the limit (in the sense of [35]) of $\mathbf{H}_\kappa$ as $\kappa \to \infty$. We use λ_i^κ, $i \in \mathbb{N}$ to denote the discrete eigenvalues of $\mathbf{H}_\kappa$, which are below the infimum of the essential spectrum and are ordered in the ascending order according to multiplicity. By $v_i^\kappa \in \mathcal{Q}(\mathfrak{h}_\kappa)$, $\mathbf{H}_\kappa v_i^\kappa = \lambda_i^\kappa v_i^\kappa$ and $\|v_i^\kappa\| = 1$ we denote accompanying eigenvectors. Here we allow $\kappa > 0$ or formally $\kappa = \infty$. Using the perturbation techniques from [13, 15, 16, 17] we prove (among other results) in the case of regular family of the type (1.1); for a definition see Section 1.2 below; the estimates

$$\frac{\text{lb}}{\kappa^2} \leq \frac{|\lambda_i^\kappa - \lambda_i^\infty|}{\lambda_i^\infty} \leq \frac{\text{ub}_1}{\kappa^2} \tag{1.4}$$

$$\frac{\text{lb}}{\kappa^2} \leq \frac{\mathfrak{h}_\kappa[v_i^\infty - v_i^\kappa]}{\mathfrak{h}_\kappa[v_i^\kappa]} \leq \frac{\text{ub}_2}{\kappa^2} \tag{1.5}$$

$$\|E_\kappa(D) - E_\infty(D)\| \leq \frac{\text{ub}_3}{\kappa^2}, \qquad D \in \mathbb{R} \setminus \operatorname{spec}(\mathbf{H}_\infty), \tag{1.6}$$

and we compute the constants lb and ub_i, $i = 1, 2, 3$ explicitly for several concrete model problems. Further, we also give a formula for determining a critical κ_0 such that (1.4)–(1.6) hold for $\kappa \geq \kappa_0$ and we show that the estimates are optimal in the sense that $\lim_{\kappa\to\infty} \frac{|\lambda_i^\kappa - \lambda_i^\infty|}{\lambda_i^\infty} \left(\frac{\text{lb}}{\kappa^2}\right)^{-1} = 1$ holds.

To show that our abstract approach to problems (1.1) does not incur accuracy tradeoffs —when applied to concrete problems—we consider several case studies. A prototype for the (less trivial) regular problem is the Arch Model from e.g. [6, Chapter 8.8:3]. In our case study we compute explicit estimates for the asymptotic behavior of the eigenvalues and spectral projections of the low frequency problem as the diameter of the arch goes to zero. The limit of such family of arches is the so called Curved Rod Model from [19, 32]. On the other hand, Schroedinger (like) operators from [2, 5, 9] are representative for (higher dimensional) operators which have less "well-behaved" spectral asymptotic. More to the point, in the case of the Schroedinger (like) operator from (1.3) we obtain the same optimality statements, but the convergence is of the fractional order $O\left(\frac{1}{\kappa^{2\alpha}}\right)$, $\alpha = \frac{1}{2}$ (cf. [9, 13] for higher dimensional problems in unbounded domains). These concrete examples determine a framework for presenting our (otherwise) more abstract results.

1.1. **Local (resolvent) estimates.** We approach this analysis by reformulating the convergence problem so that the perturbation framework and the error representation formulae (this is the main constructive feature of our framework) from [10, 13, 15, 16, 17] can be applied as a backbone of our construction. A difference between our approach and the standard results of works like [5, 7, 9, 26] can best be seen when considering a way to compute a constant ub_3 for an estimate like (1.6). The standard approach requires a study of the integral

$$\oint_{\mathfrak{C}(\lambda_i^\infty)} \left[(\zeta - \mathbf{H}_\infty)^{-1} P_{\mathsf{N}(\mathfrak{h}_e)} - (\zeta - \mathbf{H}_\kappa)^{-1}\right] d\zeta, \tag{1.7}$$

where $\mathfrak{C}(\lambda_i^\infty)$ is a circle in the resolvent set of $\mathbf{H}_\kappa$ which has λ_i^∞ in its interior and the rest of the spectrum in its exterior. This frequently leads to cumbersome estimation formulae. Thanks to the local character of the error representation formula from [13], we are able to

base our theory on a study of the integrals[1]

$$(v_i^\infty, \mathbf{H}_\kappa^{-1} v_i^\infty) - (v_i^\infty, \mathbf{H}_\infty^{-1} v_i^\infty) = \int_{\kappa^2}^\infty \|\mathbf{H}_e^{1/2} \mathbf{H}_\tau^{-1} v_i^\infty\|^2 \, d\tau, \quad i = 1, \ldots, m. \tag{1.8}$$

Here $\mathbf{H}_e$ is the operator defined by $\mathfrak{h}_e$ in the sense of Kato and $m \in \mathbb{N}$ is the multiplicity of λ_i^∞. The results from [3, 9] show that the integrals (1.8) are better amenable for a quantitative study than are (1.7).

Due to the difficulties in dealing with a formula like (1.7), typical results from semi-classical analysis from e.g. [7, 28] establish only the fact that the projections converge in a much weaker sense (than is the convergence of spectral projections in norm) without giving information on the speed of convergence as measured by the coupling κ^2. The nearest in spirit to our analysis is the approach of [26]. However, in this work only a particular family of model problems is considered and no estimates for the convergence of $E_\kappa(\cdot)$ in (unitary invariant) operator norm(s) are presented. Furthermore, the authors do not discuss the radius of convergence of their "asymptotic" expansions. For the geometric theory on the relationship between two projections and the importance of establishing convergence estimates for all unitary invariant operator norms we refer the reader to the seminal works [8, 18].

1.2. **A notion of regularity.** Let us now make precise what we mean by the regularity of $\mathfrak{h}_e$. In the terminology of [28] a family of the type (1.1) is said to be *non-inhibited stiff* if $\mathfrak{h}_e$ is a closed and positive quadratic form and the subspace

$$\mathsf{N}(\mathfrak{h}_e) := \{u \in \mathcal{Q}(\mathfrak{h}_e) \ : \ \mathfrak{h}_e[u] := \mathfrak{h}_e(u, u) = 0\} \tag{1.9}$$

(of $\mathcal{H}$) is nontrivial. For technical convenience we assume (without reducing the level of the generality) that $\mathfrak{h}_b$ is positive definite and use $\mathbf{H}_b$ and $\mathbf{H}_e$ to denote the self-adjoint operators which are defined in the sense of Kato by $\mathfrak{h}_b$ and $\mathfrak{h}_e$ respectively.

We identify the *regular family* of quadratic forms—with structure (1.1)—by requiring that $\mathfrak{h}_b$ and $\mathfrak{h}_e$ satisfy a Ladyzhenskaya–Babuška–Brezzi type condition

$$\sup_{v \in \mathcal{Q}(\mathfrak{h}_e)} \frac{|(q, \mathbf{H}_e^{1/2} v)|}{\mathfrak{h}_b[v]^{1/2}} \geq \frac{1}{\mathfrak{k}} \|P_{\mathsf{N}(\mathfrak{h}_e)} q\|, \qquad q \in \mathcal{H}, \tag{1.10}$$

for some $\mathfrak{k}$, $\mathfrak{k} > 0$. The condition (1.10) is equivalent with the claim that $\mathsf{R}(\mathbf{H}_e^{1/2} \mathbf{H}_b^{-1/2})$, the range of the operator $\mathbf{H}_e^{1/2} \mathbf{H}_b^{-1/2}$, is closed in $\mathcal{H}$, cf. examples (1.2) and (1.3). The ramifications of the assumption (1.10) will enable us to formulate a new method for studying integrals (1.8) for this class of model problems and thus complement the study of singular obstacle potentials from [3, 9].

1.3. **An outline of the paper.** Let us finish the introduction by briefly outlining the structure of the paper. In Section 2 we introduce the notation and present the qualitative convergence framework from [35]. The main approximation results of the paper appear in Section 3. To be more precise in Section 3.1 we review the operator matrix approach to Ritz value estimation from [17, 13]. In Section 3.2 this approach to spectral estimation is specialized to the problems of the large coupling limit. In particular we make precise in which sense can these estimates be considered sharp. We also revisit, in Section 3.2.1, the example from [13] to show how do (1.4)–(1.6) look in praxis for a non-regular $\mathfrak{h}_e$. In Section 4

[1] The notation $(\cdot, \cdot)$ and $\|\cdot\|$ always refer to the scalar product and the norm of the background Hilbert space $\mathcal{H}$. The functions of the operator like $\mathbf{H}_e^{1/2}$ are always meant in the sense of the spectral calculus. By $P_{\mathsf{N}(\mathfrak{h}_e)}$ we generically denote the $\mathcal{H}$ orthogonal projection onto the space $\mathsf{N}(\mathfrak{h}_e)$

we characterize regular perturbations $\mathfrak{h}_e$ and give convergence estimates which utilize this additional structural information. In Section 5 we consider a model problem from the elasticity theory and show that its asymptotic behavior is regular. In the last section we put the results in the broader context and give an outlook of further research.

At the end we would like to emphasize that our study is distinguished by its constructive character. This can be seen in the fact that we give a general method to compute the constants lb and ub_i, $i = 1, 2, 3$ (as functions of $\mathbf{H}_\kappa$ and v_i^∞) in (1.4)–(1.6). With such a result we give a method to establish both a first order correction for the limit eigenvalue λ_i^∞, as well as to assess the quality of this approximation to λ_i^κ. The optimality result is a justification of this claim. For other connections between the elementary linear algebra and spectral theory we refer the reader to [31].

2. Convergence of non-densely defined quadratic forms

In this section we fix the notation and give background information on the previous results which we use. We follow the general notational conventions and the terminology of Kato [20, Chapters VI–VIII]. Minor differences are contained in the following list of notation and terminology.

- $\mathcal{H}$ ... is an infinite dimensional Hilbert space, can be both real or complex
- $(\cdot,\cdot)$; $\|\cdot\|$ ... the scalar product on $\mathcal{H}$, linear in the second argument and anti-linear (when $\mathcal{H}$ is complex) in the first; the norm on $\mathcal{H}$
- $\mathcal{H}_1 \oplus \mathcal{H}_2$... the direct sum of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, for any $x \in \mathcal{H}_1 \oplus \mathcal{H}_2$ we have $x = x_1 \oplus x_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for $x_i \in \mathcal{H}_i$, $i = 1, 2$
- $\text{spec}(\mathbf{H})$, $\text{spec}_{ess}(\mathbf{H})$; $\lambda_{\text{ess}}(\mathbf{H})$ ... the spectrum and the essential spectrum of $\mathbf{H}$; the infimum of the essential spectrum of $\mathbf{H}$
- $A \leq B$ ... order relation between self-adjoint operators (matrices), is equivalent with the statement that $B - A$ is positive
- $\mathcal{L}(\mathcal{H})$; $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$... the space of bounded linear operators on $\mathcal{H}$, which is equipped with the norm $\|\cdot\|$; the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$
- $\mathsf{R}(X), \mathsf{N}(X)$ ... the range and the null space of the linear operator X
- $\mathbf{A}^\dagger$ ... the generalized inverse of the closed densely defined operator $\mathbf{A}$. If $\mathbf{A}$ has the closed range then $\mathbf{A}^\dagger = (\mathbf{A}(\mathbf{A}^*\mathbf{A})^{-1})^*$ is bounded, see [24]. We will extend this notion below to hold for non-densely defined self-adjoint operators.
- P, $P_\perp$... the orthogonal projections P and $P_\perp := \mathbf{I} - P$
- $j_{(\cdot)}$ ... a permutation of $\mathbb{N}$
- $\text{diag}(M, W)$ ... the block diagonal operator matrix with the operators M, W on its diagonal. The operators M, W can be both bounded and unbounded. The same notation is used to define the diagonal $m \times m$ matrix $\text{diag}(\alpha_1, \cdots, \alpha_m)$, with $\alpha_1, \cdots, \alpha_m$ on its diagonal.
- $s_1(A) \geq s_2(A) \geq \cdots$, $s_{\max}(A), s_{\min}(A)$ ... the singular values of the compact operator A ordered in the descending order according to multiplicity, the minimal (if it exists) and the maximal singular value of A
- $\|\!| X |\!\|$ ... a unitary invariant or operator cross norm of the operator X. Since $\|\!| \cdot |\!\|$ depends only on the singular values of the operator, we do not notationaly distinguish between the instances of the norm $\|\!| \cdot |\!\|$ on $\mathcal{L}(\mathcal{H})$, $\mathcal{L}(\mathsf{R}(P))$, $\mathcal{L}(\mathsf{R}(P), \mathsf{R}(P)^\perp)$, or such. For details see [30].
- $\text{tr}(X)$, $\|\!| X |\!\|_{HS}$ ... the trace a the Hilbert–Schmidt norm of the operator X, it holds $\|\!| X |\!\|_{HS} = \sqrt{\text{tr}(X^*X)}$, see [30]

As a general policy to simplify the notation we shall always drop indices when there in no danger of confusion.

Let us assume that we have a closed, symmetric and semibounded from below form $\mathfrak{h}$ with the dense domain $\mathcal{Q}(\mathfrak{h}) \subset \mathcal{H}$ as given in [20, (VI.1.5)–(VI.1.11), pp. 308–310]. The form $\mathfrak{h}$ which has a strictly positive lower bound will be called *positive-definite*. This is also a small departure from the terminology of [20, Section VI.2, pp. 310]. Such $\mathfrak{h}$ defines the self-adjoint and positive definite operator $\mathbf{H}$ in the sense of [20, Theorem VI.2.23, pp. 331]. Furthermore, the operator $\mathbf{H}$ is densely defined with the domain $\mathcal{D}(\mathbf{H}) \subset \mathcal{Q}(\mathfrak{h})$ and $\mathcal{D}(\mathbf{H}^{1/2}) = \mathcal{Q}(\mathfrak{h})$. We also generically assume that $\mathbf{H}$ has discrete eigenvalues $\lambda_1(\mathbf{H}) \leq \cdots \leq \lambda_m(\mathbf{H}) \leq \cdots < \lambda_{\text{ess}}(\mathbf{H})$, where we count the eigenvalues according to multiplicity. Another departure from the terminology of Kato is that we use $\mathfrak{h}(\psi, \phi)$ to denote the value of $\mathfrak{h}$ on $\psi, \phi \in \mathcal{Q}(\mathfrak{h})$, but we write $\mathfrak{h}[\psi] := \mathfrak{h}(\psi, \psi)$ for the associated *quadratic form* $\mathfrak{h}[\cdot]$. We also emphasize that we use $\cdot^*$ to denote the adjoint both in the real as well as in the complex Hilbert space $\mathcal{H}$ as is customary in [20, Chapters VI–VIII].

In order to be able to handle the problems of the type (1.1), we shall need to work with operators that are not necessarily densely defined, cf. (1.2) and (1.3). We use the notion of the *pseudo inverse* of the operator $\mathbf{H}$ that is assumed to be self-adjoint in the closure of its domain of definition $\overline{\mathcal{D}(\mathbf{H})}^{\|\cdot\|} \subset \mathcal{H}$ (tacitly assumed to be a non-trivial subspace). A definition from [35] will be used. The *pseudo inverse* of the operator $\mathbf{H}$ is the self-adjoint operator $\widehat{\mathbf{H}}$ defined by

$$\mathcal{D}(\widehat{\mathbf{H}}) = \mathsf{R}(\mathbf{H}) \oplus \mathcal{D}(\mathbf{H})^{\perp},$$
$$\widehat{\mathbf{H}}(u + v) = \mathbf{H}^{-1} u, \qquad u \in \mathsf{R}(\mathbf{H}),\ v \in \mathcal{D}(\mathbf{H})^{\perp}.$$

It follows that $\widehat{\mathbf{H}} = \mathbf{H}^{-1}$ in $\overline{\mathsf{R}(\mathbf{H})}^{\|\cdot\|}$ and $\widehat{\mathbf{H}}$ is bounded if and only if $\mathsf{R}(\mathbf{H})$ is closed in $\mathcal{H}$. When considered solely in $\overline{\mathcal{D}(\mathbf{H})}^{\|\cdot\|}$ the operator $\mathbf{H}$ is obviously self-adjoint, so we can also use the spectral calculus from [29] to define the *generalized inverse*, which extends the definition from the case of the densely defined operator, as

$$\mathbf{H}^{\dagger} = f(\mathbf{H}), \qquad f(\lambda) = \begin{cases} 0, & \lambda = 0 \\ \frac{1}{\lambda}, & \lambda > 0 \end{cases}$$
$$\mathcal{D}(\mathbf{H}^{\dagger}) = \{u \in \mathcal{H} \,:\, \int f^2(\lambda) d(E(\lambda)u, u) < \infty\},$$

where $E(\cdot) = E_{\mathbf{H}}(\cdot) P_{\mathcal{D}(\mathbf{H})}$. Obviously, we have $\mathcal{D}(\widehat{\mathbf{H}}) \oplus \mathsf{N}(\mathbf{H}) = \mathcal{D}(\mathbf{H}^{\dagger})$ and the identity $\mathbf{H}^{\dagger} u = \widehat{\mathbf{H}} u$, $u \in \mathcal{D}(\widehat{\mathbf{H}}^{1/2})$ holds. In further text we shall tacitly drop the notational distinction between the generalized and pseudo inverse. The usual monotonicity properties can be extended to the generalized inverse. In particular it holds

$$\|\mathbf{H}_1^{1/2} u\| \leq \|\mathbf{H}_2^{1/2} u\|,\ \ u \in \mathcal{D}(\mathbf{H}_2^{1/2}) \Leftrightarrow \|\mathbf{H}_2^{1/2\dagger} u\| \leq \|\mathbf{H}_1^{1/2\dagger} u\|,\ \ u \in \mathcal{D}(\widehat{\mathbf{H}}_1^{1/2}). \tag{2.1}$$

This monotonicity principle is the main ingredient of the proof of the convergence result for (1.1). When dealing with non-densely defined forms this principle can be formulated as follows. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two closed positive definite forms and let $\mathbf{H}_1$ and $\mathbf{H}_2$ be the self-adjoint operators defined by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ in $\overline{\mathcal{Q}(\mathfrak{h}_1)}$ and $\overline{\mathcal{Q}(\mathfrak{h}_2)}$. We say $\mathfrak{h}_1 \leq \mathfrak{h}_2$ when $\mathcal{Q}(\mathfrak{h}_2) \subset \mathcal{Q}(\mathfrak{h}_1)$ and

$$\mathfrak{h}_1[u] = \|\mathbf{H}_1^{1/2} u\|^2 \leq \mathfrak{h}_2[u] = \|\mathbf{H}_2^{1/2} u\|^2, \qquad u \in \mathcal{Q}(\mathfrak{h}_2). \tag{2.2}$$

Equivalently, we write $\mathbf{H}_1 \leq \mathbf{H}_2$ when $\mathfrak{h}_1 \leq \mathfrak{h}_2$. Now, we can write the fact (2.1) as

$$\mathbf{H}_1 \leq \mathbf{H}_2 \Longleftrightarrow \mathbf{H}_2^{\dagger} \leq \mathbf{H}_1^{\dagger}. \tag{2.3}$$

Let us define, for non-inhibited (see definition (1.9)) quadratic forms like $\mathfrak{h}_\kappa$ from (1.1), the domain $\mathcal{Q}_\infty := \{u \in \mathcal{Q} \ : \ \lim_{\kappa\to\infty} \mathfrak{h}_\kappa[u] < \infty\}$, then according to [29, 35] the symmetric form

$$\mathfrak{h}_\infty(u,v) = \lim_{\kappa\to\infty} \mathfrak{h}_\kappa(u,v), \qquad u,v \in \mathcal{Q}_\infty$$

is closed in $\overline{\mathcal{Q}_\infty}^{\|\cdot\|}$ and it defines the self-adjoint operator $\mathbf{H}_\infty$ there. Further, it holds that $\mathbf{H}_\infty^\dagger = \text{s-}\lim_{k\to\infty} \mathbf{H}_\kappa^{-1}$. The general framework for a description of families of converging positive definite forms will be the following theorem from [35].

Theorem 2.1. *Let $\mathfrak{s}_n$, $\mathfrak{h}_n$, $\mathfrak{u}_n$ and $\mathfrak{h}_\infty$ be closed symmetric forms in $\mathcal{H}$ such that they are all uniformly*[2] *positive definite.*

(1) *If $\mathfrak{s}_n \geq \mathfrak{s}_{n+1} \geq \mathfrak{h}_\infty$ where*

$$\mathfrak{h}_\infty(u,v) = \lim_{n\to\infty} \mathfrak{s}_n(u,v), \qquad u,v \in \bigcup_{n\in\mathbb{N}} \mathcal{Q}(\mathfrak{s}_n)$$

then $\mathfrak{h}_\infty$ is closed with $\mathcal{Q}(\mathfrak{h}_\infty) = \overline{\bigcup_{n\in\mathbb{N}} \mathcal{Q}(\mathfrak{s}_n)}^{\,\mathfrak{h}_\infty}$ and $\mathbf{H}_\infty^\dagger = \text{s-}\lim_n \mathbf{S}_n^\dagger$.

(2) *If $\mathfrak{u}_n \leq \mathfrak{u}_{n+1} \leq \mathfrak{h}_\infty$ where*

$$\mathfrak{h}_\infty(u,v) = \lim_{n\to\infty} \mathfrak{u}_n(u,v), \qquad u,v \in \mathcal{Q}(\mathfrak{h}_\infty)$$

then $\mathfrak{h}_\infty$ is closed with $\mathcal{Q}(\mathfrak{h}_\infty) = \left\{f \in \bigcap_{n\in\mathbb{N}} \mathcal{Q}(\mathfrak{u}_n) \ : \ \sup \mathfrak{u}_n[f] < \infty\right\}$ and $\mathbf{H}_\infty^\dagger = \text{s-}\lim_n \mathbf{U}_n^\dagger$.

(3) *If $\mathfrak{u}_n$ and $\mathfrak{s}_n$ are as before and $\mathfrak{u}_n \leq \mathfrak{h}_n \leq \mathfrak{s}_n$ also holds, then*

$$\mathfrak{h}_\infty(u,v) = \lim_{n\to\infty} \mathfrak{h}_n(u,v), \qquad u,v \in \mathcal{Q}(\mathfrak{h}_\infty),$$
$$\mathbf{H}_\infty^\dagger = \underset{\kappa\to\infty}{\text{s-lim}}\, \mathbf{H}_n^\dagger.$$

For the families of forms which satisfy the assumptions of Theorem 2.1 the following qualitative convergence result on spectral families has been established in [35].

Theorem 2.2. *Let $\mathfrak{h}_n$ be a sequence of positive definite forms that satisfies any of the assumptions of Theorem 2.1 for $\mathfrak{u}_n$, $\mathfrak{s}_n$ or $\mathfrak{h}_n$. Let there also be the positive definite form $\mathfrak{s}$ such that $\mathfrak{h}_n \geq \mathfrak{s}$ and $\lambda_e(\mathbf{S}) > 0$. Then*

$$\|E_n(D) - E_\infty(D)\| \to 0, \qquad D < \lambda_e(\mathbf{S}), D \notin \operatorname{spec}(\mathbf{H}_\infty). \tag{2.4}$$

The results like Theorem 2.1 have independently been obtained in [28, 29]. We have opted for Theorem 2.1 since it extensively uses the monotonicity (or “sandwiched” monotonicity) to establish the stability of the converging eigenvalues and this fits neatly into the perturbation framework of [10]. This was the chief source of motivation for the main construction from the PhD thesis [14] (those results appeared later in [13, 15, 16, 17]).

3. A CONSTRUCTIVE APPROACH TO ASYMPTOTIC EIGENVALUE/EIGENVECTOR ESTIMATES

Let us reiterate that we use the notion of the constructiveness in this paper in two contexts. First, it should emphasize that all of our theory is bases on the error representation result like (3.8)–(3.9), below. But second, it is also meant to emphasize that in a result like those of the type (1.4)–(1.6) we present a way to construct an improvement to the approximation λ_i^∞ (of the eigenvalue λ_i^κ). The constants lb and ub$_i$, $i = 1,2,3$ are explicit functions of the approximation defects $\eta_i(P)$, to be defined below and it is the aim of this section to reveal this dependence.

[2] By this we mean that they have a uniform positive lower bound.

3.1. **Background information on the block-diagonal part of the operator/form.** In this section we review the results form our previous work which we use to prove our first contribution in Section 3.2. A reader who would like to go straight to the new results can do that directly after reading equation (3.1) and Definition 3.1 below.

In this section we assume that we have a fixed closed symmetric and densely defined form $\mathfrak{h}$. We will review the basic spectral properties of the *block-diagonal part* of $\mathfrak{h}$ with respect to orthogonal projection P, $\mathsf{R}(P) \subset \mathcal{Q}(\mathfrak{h})$ as is presented in [15]. In order to simplify the presentation we temporarily suppress (in the notation) the dependence of quantities on $\mathbf{H}$, where there is no danger of confusion. Assuming that $\mathsf{R}(P)$ is finite dimensional we define the *block-diagonal* part of $\mathfrak{h}$ by setting

$$\mathfrak{h}_P(u,v) := \mathfrak{h}(Pu,Pv) + \mathfrak{h}(P_\perp u, P_\perp v), \qquad u,v \in \mathcal{Q}(\mathfrak{h}_P) := \mathcal{Q}(\mathfrak{h}). \tag{3.1}$$

Obviously the form $\mathfrak{h}_P$ is closed and positive definite and so it defines the self-adjoint operator $\mathbf{H}_P$ in the sense of Kato. We further have (for a proof see [13, 15]):

$$\mathsf{R}(\mathbf{H}^{-1} - \mathbf{H}_P^{-1}) \qquad \text{is finite dimensional.} \tag{3.2}$$

$$\eta_{\max}(P) := \sup_{u \in \mathcal{Q}(\mathfrak{h})} \frac{|\mathfrak{h}[u] - \mathfrak{h}_P[u]|}{\mathfrak{h}_P[u]} < 1. \tag{3.3}$$

A first consequence of these two features is the stability of essential spectra, namely Weyl's theorem gives $\operatorname{spec}_{\text{ess}}(\mathbf{H}) = \operatorname{spec}_{\text{ess}}(\mathbf{H}_P)$. Further, we have the estimate—of the same form as (3.3)— for the eigenvalues $\lambda_i(\mathbf{H}_P)$ and $\lambda_i(\mathbf{H})$, $i \in \mathbb{N}$ which are below the infimum of the essential spectrum $\lambda_{\text{ess}}(\mathbf{H}) = \lambda_{\text{ess}}(\mathbf{H}_P)$

$$\frac{|\lambda_i(\mathbf{H}) - \lambda_i(\mathbf{H}_P)|}{\lambda_i(\mathbf{H}_P)} < \eta_{\max}(P), \qquad i \in \mathbb{N}. \tag{3.4}$$

The attractiveness of interpreting the form $\mathfrak{h}$ as a perturbation of its block-diagonal part lies in the fact that

$$\operatorname{spec}(\mathbf{H}_P) = \operatorname{spec}(\Xi) \cup \operatorname{spec}(\mathbf{W}) \tag{3.5}$$

where $\Xi = (\mathbf{H}^{1/2}P)^*(\mathbf{H}^{1/2}P)\big|_{\mathsf{R}(P)}$ is a finite dimensional operator and $\mathbf{W}$ is the self-adjoint operator which is defined in $\mathsf{R}(P_\perp)$ by the quadratic form $\mathfrak{h}(P_\perp\cdot, P_\perp\cdot)$. Since $\operatorname{spec}(\Xi)$ is computable, we can start building our constructive estimation procedure on this fact. As a convention we will use $\mu_1 \le \cdots \le \mu_{\dim \mathsf{R}(P)}$ to denote the eigenvalues of Ξ. The numbers μ_i will be called the *Ritz values* from the subspace $\mathsf{R}(P)$. In this section we also use the notation $\lambda_i := \lambda_i(\mathbf{H})$.

Let us now assume that $\dim \mathsf{R}(P) = m \in \mathbb{N}$. To examine the relationship between $\mathfrak{h}$ and $\mathfrak{h}_P$ in further detail define

$$\eta_i(P) := \Big[\max_{\substack{\mathcal{S} \subset \mathsf{R}(P), \\ \dim(\mathcal{S}) = m-i+1}} \min \Big\{ \frac{(\psi, \mathbf{H}^{-1}\psi) - (\psi, \mathbf{H}_P^{-1}\psi)}{(\psi, \mathbf{H}^{-1}\psi)} \;\Big|\; \psi \in \mathcal{S}, \|\psi\| = 1 \Big\} \Big]^{1/2}, \tag{3.6}$$

for $i = 1, \ldots, m$. It has also been shown in [15] that $\eta_{\max}(P) = \eta_m(P)$. Although the perturbation $\delta_P(\mathfrak{h}) := \mathfrak{h} - \mathfrak{h}_P$ is in general—for some P, $\mathsf{R}(P) \subset \mathcal{Q}(\mathfrak{h})$—not representable by an operator, the quadratic form $\delta_P^s(\mathfrak{h})[\cdot] := \mathfrak{h}[\mathbf{H}_P^{-1/2}\cdot] - \mathfrak{h}_P[\mathbf{H}_P^{-1/2}\cdot]$ can always be represented by the bounded operator block-matrix (with respect to $P \oplus P_\perp = I$)

$$\delta_P^s(H) = \begin{pmatrix} 0 & \Gamma^* \\ \Gamma & 0 \end{pmatrix}, \qquad \text{and} \quad (\cdot, \delta_P^s(H)\cdot) = \delta_P^s(\mathfrak{h})[\cdot].$$

Furthermore, [13, Lemma 2.1] gives that $s_i(\Gamma) = \eta_i(P)$, $i = 1, \ldots, m$. The analysis of [13] now yields the conclusion that the test space $\mathsf{R}(P)$ can be used to generate good

approximation for the eigenvalues λ_i, $i = q, \ldots, q+m-1$ when $\eta_m(P)$ is smaller than half of the *relative gap*

$$\gamma_q := \min\Big\{\frac{\lambda_{q+m}-\mu_m}{\lambda_{q+m}+\mu_m}, \frac{\mu_1-\lambda_{q-1}}{\mu_1+\lambda_{q-1}}\Big\}.$$

Ample numerical evidence corroborate that such estimates are robust (with regard to scaling) and sharp. Assume that $\eta_{\max}(P) < \frac{1}{2}\gamma_q$ and that $\dim \mathsf{R}(P) = m$, where m is the multiplicity of the eigenvalue λ_q. Using [13, Theorem 3.3] we conclude that the operator matrix

$$\delta_P^s(H_q) = \begin{bmatrix} \mathbf{I}-\lambda_q\Xi^{-1} & \Gamma^* \\ \Gamma & \mathbf{I}-\lambda_q\mathbf{W}^{-1} \end{bmatrix}, \tag{3.7}$$

which is the block-matrix representation (with respect to $P \oplus P_\perp = \mathbf{I}$) of the quadratic form

$$\delta_P^s(\mathfrak{h}_q)[\cdot] := \mathfrak{h}(\mathbf{H}_P^{-1/2}\cdot, \mathbf{H}_P^{-1/2}\cdot) - \lambda_q(\mathbf{H}_P^{-1/2}\cdot, \mathbf{H}_P^{-1/2}\cdot),$$

satisfies $\dim \mathsf{N}(\delta_P^s(H_q)) = m$ and the mechanism of [31, (1.1)–(1.2)]—also known in the Linear Algebra as the Wilkinson's Schur complement trick (see [27, pp. 183] and [13, Theorem 3.3])—allows us to conclude

$$\mathbf{I}-\lambda_q\Xi^{-1} = \Gamma^*(\mathbf{I}-\lambda_q\mathbf{W}^{-1})^{-1}\Gamma \tag{3.8}$$

$$= \Gamma^*\Gamma + \lambda_q\Gamma^*\mathbf{W}^{-1/2}(\mathbf{I}-\lambda_q\mathbf{W}^{-1})^{-1}\mathbf{W}^{-1/2}\Gamma. \tag{3.9}$$

Identity (3.8) is the basis of the proof of [13, Theorem 3.3] which we now quote. Note that (3.8)–(3.9) also hold for λ_q which is in a gap of the essential spectrum. Based on the definition (3.6) we now define (for later usage) the *approximation-defects* for $\mathfrak{h}_\kappa$.

Definition 3.1. Let the sequence $\mathfrak{h}_\kappa$ be given and let the orthogonal projection P be such that $\mathsf{R}(P) \subset \mathcal{Q}(\mathfrak{h}_\kappa)$ and $\dim \mathsf{R}(P) < \infty$. We write $\eta_i(\kappa, P)$ for $\eta_i(P)$ from (3.6) when applied on $\mathfrak{h}_\kappa$. We call $\eta_i(\kappa, P)$ the *κ-approximation defects*. If we are given a subspace $\mathfrak{P} = \mathsf{R}(P)$, then we abuse (simplify) the notation and freely write $\eta_i(\kappa, \mathfrak{P}) = \eta_i(\kappa, P)$.

Theorem 3.2. *Let the discrete eigenvalues of the positive definite operator $\mathbf{H}$ be so ordered that $\lambda_{q-1} < \lambda_q = \lambda_{q+m-1} < \lambda_{q+m}$. Let $\mathsf{R}(P) \subset \mathcal{Q}(h)$ be the test subspace such that $\dim \mathsf{R}(P) = m$ and $\frac{\eta_m(P)}{1-\eta_m(P)} < \gamma_q$. Then we have*

$$\| \operatorname{diag}(\frac{|\lambda_q-\mu_i|}{\mu_i})_{i=1}^m \| \le \frac{\eta_m(P)}{\mathfrak{g}_{q,\eta_m(P)}} \| \operatorname{diag}(\eta_i(P))_{i=1}^m \| . \tag{3.10}$$

where $\mathfrak{g}_{q,\zeta} := \max\big\{\frac{\mu_1(1-\zeta)-(1+\frac{\zeta}{1-\zeta})\lambda_{q-1}}{(1+\frac{\zeta}{1-\zeta})\lambda_{q-1}}, \frac{(1-\frac{\zeta}{1-\zeta})\lambda_{q+m}-(1+\zeta)\mu_m}{(1-\frac{\zeta}{1-\zeta})\lambda_{q+m}}\big\}$ for $q > 1$ and we set $\mathfrak{g}_{1,\zeta} := \mathfrak{g}_1 := \frac{\lambda_{m+1}-\mu_m}{\lambda_{m+1}+\mu_m}$. Here we use $\operatorname{diag}(\alpha_i)_{i=1}^m$ to denote the $m \times m$ diagonal matrix with scalars α_i on its diagonal and $\| \cdot \|$ denotes any unitary invariant matrix norm and μ_i are the Ritz values from $\mathsf{R}(P)$.

In the case in which we do not have explicit information on the multiplicity of λ_q we have a weaker upper estimate. There is also an accompanying lower estimate which establishes the equivalence of the estimators $\eta_i(P)$ and the error. Assuming that $\mathbf{H} = \int \lambda\, dE(\lambda)$ and that we use v_i, $\mathbf{H}v_i = \lambda_i v_i$, $\|v_i\| = 1$ to denote eigenvectors and $\psi_i \in \mathsf{R}(P)$, $\Xi\psi_i = \mu_i\psi_i$, $\|\psi_i\| = 1$ to denote Ritz vectors, we collect some representative spectral estimates (bases on $\mathsf{R}(P)$) from [13, 17].

Theorem 3.3. *Let the discrete eigenvalues of the positive definite operator* $\mathbf{H}$ *be so ordered that* $\lambda_m < \lambda_{m+1}$ *and let* $\lambda_{s_1} < \lambda_{s_2} < \cdots < \lambda_{s_p}$ *be all the elements*[3] *of* $\mathrm{spec}(\mathbf{H}) \setminus \{\lambda \in \mathrm{spec}(\mathbf{H}) \;:\; \lambda \geq \lambda_{m+1}\}$. *If* $\frac{\eta_m(P)}{1-\eta_m(P)} < \frac{\lambda_{m+1}-\mu_m}{\lambda_m+\mu_m}$ *then there exist eigenvectors* v_i, $\mathbf{H}v_i = \lambda_i v_i$, $\|v_i\| = 1$ *and Ritz vectors* $\psi_i \in \mathsf{R}(P)$, $\Xi\psi_i = \mu_i\psi_i$, $\|\psi_i\| = 1$ *such that*

$$\| E(\mu_m) - P \| \leq \frac{\sqrt{\lambda_{m+1}\mu_m}}{\lambda_{m+1}-\mu_m} \frac{\| \mathrm{diag}((\eta_i(P))_{i=1}^m) \oplus \mathrm{diag}((\eta_i(P))_{i=1}^m) \|}{\sqrt{1-\eta_m(P)}}, \tag{3.11}$$

$$\frac{\mu_1}{2\mu_m}\sum_{i=1}^m \eta_i^2(P) \leq \sum_{i=1}^m \frac{|\lambda_i - \mu_i|}{\mu_i} \leq \frac{1}{\min\limits_{i=1,\dots,p} \mathfrak{g}_{s_i,\eta_{m_i}(P_{s_i})}} \sum_{i=1}^m \eta_i^2(P), \tag{3.12}$$

$$\|v_i - \psi_i\| \leq \max_{\lambda\in\mathrm{spec}(\mathbf{H})\setminus\{\lambda_i\}} \frac{\sqrt{2\lambda\mu_i}}{|\lambda-\mu_i|} \frac{\eta_m(P)}{\sqrt{1-\eta_m(P)}}, \tag{3.13}$$

$$\frac{\mathfrak{h}[\psi_i - v_i]}{\mathfrak{h}[v_i]} = \|v_i - \psi_i\|^2 + \frac{\mu_i - \lambda_i}{\lambda_i}, \qquad i = 1, ..., m. \tag{3.14}$$

Here P_{s_i} *is the orthogonal projection onto the linear span of* $\{\psi_j \;:\; j = \sum_{k=1}^i m_k + 1, \dots, \sum_{k=1}^{i+1} m_k\}$ *and* m_i *is the multiplicity of the eigenvalue* λ_{s_i}, $i = 1, \dots, p$. *Obviously the identity* $P_{s_1} \oplus P_{s_2} \oplus \cdots \oplus P_{s_p} = P$ *holds. In the case in which* $\lambda_1 = \lambda_m$ *we can drop the constant* $\frac{\mu_1}{2\mu_m}$ *from the lower estimate. We can also allow for other cross norms* $\| \cdot \|$ *of the diagonal matrix* $\mathrm{diag}((\eta_i(P))_{i=1}^m)$ *in (3.12).*

The proof of the estimate for the spectral projection (3.11) can be found in [17], the proof of (3.13) is in [13] and identity (3.14) is well-known. For reader's convenience let us also point out that the problem of estimating the spectral projections $E(\mathfrak{I})$—where $\mathfrak{I}$ is some contiguous interval whose boundary points are not the accumulation points of $\mathrm{spec}(\mathbf{H})$— can be seen as problem in obtaining a robust computable estimate of the Cauchy integral

$$\|E(\mathfrak{I}) - P\| = \frac{1}{2\pi}\| \oint_{\mathfrak{C}(\mathfrak{I})} (\zeta - \mathbf{H}_P)^{-1} - (\zeta - \mathbf{H})^{-1}\, d\zeta\|. \tag{3.15}$$

By $\mathfrak{C}(\mathfrak{I})$ we denote the circle in the resolvent set of $\mathbf{H}$ such that $\mathfrak{I}$ is in the interior of the associated disc and the rest of the spectrum is outside the disc. However, contrary to the intuition, the direct analysis of (3.15) is not the most natural way to obtain computable and robust estimates of $\|E(\mathfrak{I}) - P\|$. A problem is that, although the integral of the resolvent difference does not depend on the integration path $\mathfrak{C}(\mathfrak{I})$, estimates of it do. Furthermore, the circle is only one of many possible curves which should be taken into account. As an alternative we consider the approach of the (weakly formulated) operator equations. Not only are the estimation formula which are so obtained sharp (see [17, Remark 2.3]), but also the technique allows for a natural consideration of estimates which utilize other operator cross norms $\| \cdot \|$. Such results are known as $\sin\Theta$ theorems in the recognition of the milestone work [8] and have been extensively studied in the computational Linear Algebra, see [22, 21] and the references there. We use a recent generalization of those results, which is particularly suitable for an application in the quadratic form setting, see [17].

[3] We assume that $1 \leq s_1 < s_2 < \cdots < s_p \leq m$.

Remark 3.4. Note that as $\eta_{s_i}(P_i) \to 0$ we have $\mathfrak{g}_{s_i,\eta_{m_i}(P_i)} \to \min\{\frac{\lambda_{s_{i+1}}-\lambda_{s_i}}{\lambda_{s_i}}, \frac{\lambda_{s_i}-\lambda_{s_{i-1}}}{\lambda_{s_{i-1}}}\}$ and $\min\{\mathfrak{g}_{s_i,\eta_{m_i}(P_i)} \,:\, i = 1,\dots,p\}$ quantifies the minimal *relative* gap among the eigenvalues $\lambda_{s_1} < \lambda_{s_2} < \cdots < \lambda_{s_p}$. Note that the relative gap $\mathfrak{g}_{s_i,\eta_{s_i}(P_i)}$ distinguishes better between the close eigenvalues than the *absolute* gap, eg. $\min\{\lambda_{s_{i+1}} - \lambda_{s_i}, \lambda_{s_i} - \lambda_{s_{i-1}}\}$ is an example of an absolute gap. In Theorem 3.3, equivalently as in [11, Proposition 2.3], we have that when $\eta_{m_i}(P_i) < \frac{1}{3}\min_{k\neq j}\frac{|\lambda_{s_k}-\lambda_{s_j}|}{\lambda_{s_k}+\lambda_{s_j}}$, $i = 1,\dots,p$ then

$$\frac{1}{\min\limits_{i=1,\dots,p} \mathfrak{g}_{s_i,\eta_{m_i}(P_i)}} \leq \frac{3}{\min_{k\neq j}\frac{|\lambda_{s_k}-\lambda_{s_j}|}{\lambda_{s_k}+\lambda_{s_j}}}.$$

3.2. **Estimates for the spectral asymptotic.** We will now use Theorem 3.3 to obtain convergence rate estimates for (2.4). This is the central result which guaranties the stability of the spectrum of the converging family of forms $\mathfrak{h}_\kappa$. Subsequently we will also prove results like (1.4)–(1.6) and use the motivating example of the Schroedinger operator with a singular obstacle potential from [13, Section 4] to show our estimates in action.

Although we are working under the assumptions of Theorem 2.1, we assume—in order to be more explicit— that we have the non-inhibited stiff family $\mathfrak{h}_\kappa$ from (1.1). The form $\mathfrak{h}_\infty$ obviously defines the self-adjoint operator $\mathbf{H}_\infty$ in $\mathsf{N}(\mathfrak{h}_e)$. By $\mathbf{H}_\infty = \int \lambda E_\infty(\lambda)$ we denote the spectral representation of $\mathbf{H}_\infty$ in $\mathsf{N}(\mathfrak{h}_e)$. We identify $E_\infty(\cdot)$ with $E_\infty(\cdot)P_{\mathsf{N}(\mathfrak{h}_e)}$ and write $\mathbf{H}_\infty = \int \lambda E_\infty(\lambda)$ for the non-densely defined—in the space $\mathcal{H}$—operator $\mathbf{H}_\infty$. Let $\mathfrak{I}$ be a contiguous interval in $\mathbb{R}$, then $\mathfrak{E}_\infty^{\mathfrak{I}} := \mathsf{R}(E_\infty(\mathfrak{I}))$ is a subspace of $\mathcal{Q} := \mathcal{Q}(\mathfrak{h}_b)$. Let now $\mathfrak{I}$ be such that $\mathfrak{E}_\infty^{\mathfrak{I}}$ is finite dimensional, then κ-approximation defect is given by

$$\eta_i(\kappa, \mathfrak{E}_\infty^{\mathfrak{I}}) := \Big[\max_{\substack{\mathcal{S}\subset\mathfrak{E}_\infty,\\ \dim(\mathcal{S})=m-i+1}} \min\Big\{\frac{(\psi, \mathbf{H}_\kappa^{-1}\psi) - (\psi, \mathbf{H}_{\mathfrak{E}_\infty}^{-1}\psi)}{(\psi, \mathbf{H}_\kappa^{-1}\psi)} \;\Big|\; \psi \in \mathcal{S}, \|\psi\| = 1\Big\}\Big]^{1/2}, \tag{3.16}$$

where $\mathbf{H}_{\mathfrak{E}_\infty^{\mathfrak{I}}}^{-1} := (\mathbf{H}_\infty^\dagger)_{E_\infty(\mathfrak{I})} = (\mathbf{H}_\kappa^{-1})_{E_\infty(\mathfrak{I})}$ and $i = 1,\dots,\dim\mathsf{R}(\mathfrak{E}_\infty^{\mathfrak{I}})$. To further simplify the notation we set $\eta_i(\kappa,\mathfrak{I}) := \eta_i(\kappa, \mathfrak{E}_\infty^{\mathfrak{I}})$. Theorem 2.1 now obviously yields

$$\lim_{\kappa\to\infty} \eta_i(\kappa,\mathfrak{I}) = 0, \qquad i = 1,\dots,\dim\mathsf{R}(\mathfrak{E}_\infty^{\mathfrak{I}}).$$

Similar construction can be performed in the case in which $\mathfrak{E}_\infty^{\mathfrak{I}}$ is infinite dimensional. The main features which are lost in this generalization are the easy computability of $\mathrm{spec}(\Xi_\kappa^{-1}) = \mathrm{spec}(\mathbf{H}_{\mathfrak{E}_\infty^{\mathfrak{I}}}^{-1} E_\infty(\mathfrak{I}))$, the property that always $\eta_{\max}(\kappa,\mathfrak{I}) < 1$ and the result on the stability of the essential spectrum. This makes, in general, such method less attractive for practical constructive considerations.

Let us first give a quantitative version of Theorem 2.1 which is based on the application of Theorem 3.3. As a notational convenience we use $\lambda_1^\infty \leq \cdots \leq \lambda_i^\infty \leq \lambda_{\mathrm{ess}}^\infty$ and $\lambda_1^\kappa \leq \cdots \leq \lambda_i^\kappa \leq \lambda_{\mathrm{ess}}^\kappa$ to denote the discrete eigenvalues below the infimum of the essential spectrum of the operators $\mathbf{H}_\infty$ and $\mathbf{H}_\kappa$ respectively.

Theorem 3.5. *Let $\mathbf{H}_\kappa = \int \lambda \; d\; E_\kappa(\lambda)$ be the operators which are associated with the family of forms $\mathfrak{h}_\kappa$. Take $D \in \mathbb{R}$ such that $\lambda_m^\infty < D < \lambda_{m+1}^\infty$ and set $\mathfrak{I} = \langle -\infty, D]$, then*

$$\eta_i(\kappa,\mathfrak{I}) < 1, \qquad i = 1,\dots,m, \tag{3.17}$$

$$\frac{|\lambda_j^\kappa - \lambda_j^\infty|}{\lambda_j^\infty} \leq \eta_m(\kappa,\mathfrak{I}), \quad j = 1,\dots,m, \tag{3.18}$$

$$\|E_\kappa(D) - E_\infty(D)\| \leq \frac{\sqrt{D\lambda_m^\infty}}{|D - \lambda_m^\infty|}\frac{\eta_m(\kappa,\mathfrak{I})}{\sqrt{1-\eta_m(\kappa,\mathfrak{I})}} \tag{3.19}$$

for κ large enough. (For the meaning of the phrase large enough see Remark 3.6.)

Proof. Statement (3.17) is a direct consequence of [13, Lemma 2.1]. Let us now remember (3.4). This estimate is the consequence of [15, Theorem 4.5] which, when applied to the form $\mathfrak{h}_\kappa$ and its $\mathfrak{E}_\infty$ block-diagonal part $(\mathfrak{h}_\kappa)_{\mathfrak{E}_\infty}$, yields

$$(1-\eta_m(\kappa,\mathfrak{I}))\big(\mathfrak{h}_\kappa\big)_{\mathfrak{E}_\infty} \leq \mathfrak{h}_\kappa \leq (1+\eta_m(\kappa,\mathfrak{I}))\big(\mathfrak{h}_\kappa\big)_{\mathfrak{E}_\infty}.$$

Let $\big(\mathbf{H}_\kappa\big)_{\mathfrak{E}_\infty}$ be the self-adjoint operators which represent the forms $\big(\mathfrak{h}_\kappa\big)_{\mathfrak{E}_\infty}$ in the sense of Kato, then $\lambda_i^\infty \in \operatorname{spec}\big(\mathbf{H}_\kappa\big)_{\mathfrak{E}_\infty}$. Set $\mathbf{W}_\kappa = \big(\mathbf{H}_\kappa\big)_{\mathfrak{E}_\infty} E_\infty(D)_\perp$ and $\mathbf{W}_\infty = \mathbf{H}_\infty E_\infty(D)_\perp$ then Theorem 2.1 implies that $\mathbf{W}_\infty^\dagger = \text{s-lim}_{\kappa\to\infty}\mathbf{W}_\kappa^\dagger$. By the construction of $\mathbf{W}_\kappa$ we have $\mathsf{R}(E_\infty(D)) \perp w$ for any $w \in \mathcal{D}(\mathbf{W}_\kappa)$. This implies $\lambda_1(\mathbf{W}_\kappa) \to \lambda_1(\mathbf{W}_\infty) = \lambda_{m+1}^\infty$. On the other hand, since

$$\operatorname{spec}(\big(\mathbf{H}_\kappa\big)_{\mathfrak{E}_\infty}) = \{\lambda \in \operatorname{spec}(\mathbf{H}_\infty) \;:\; \lambda \leq D\} \cup \operatorname{spec}\mathbf{W}_\kappa$$

it follows that there is κ_0 such that

$$\big[\lambda_m^\infty, D\big] \subset \mathbb{R} \setminus \operatorname{spec}(\big(\mathbf{H}_\kappa\big)_{\mathfrak{E}_\infty}), \qquad \kappa > \kappa_0.$$

Since $\eta_m(\kappa,\mathfrak{I}) \to 0$ we conclude that for $\kappa > \kappa_0$ (here we slightly abuse the notation) the estimate $\eta_m(\kappa,\mathfrak{I}) \leq \frac{1}{2}\frac{D-\lambda_m^\infty}{D+\lambda_m^\infty}$ holds. Now, the conclusion (3.18) follows from [15, Theorem 5.2]. Equivalently, the conclusion (3.19) follows from (3.11) and [17, Theorem 3.2]. □

Remark 3.6. The coupling constant κ_0 is large enough when

$$\eta_m(\kappa,\mathfrak{I}) < \frac{1}{3}\frac{\lambda_{m+1}^\infty - \lambda_m^\infty}{\lambda_{m+1}^\infty + \lambda_m^\infty}$$

for $\kappa > \kappa_0$. This follows by a similar consideration as in Remark 3.4.

A direct application of the results from [17, Section 3] and the results of Theorem 3.5 is the following corollary.

Corollary 3.7. *Assuming the setting and the notation of the previous theorem we have*

$$\| E_\kappa(D) - E_\infty(D) \| \leq \frac{\sqrt{D\lambda_m^\infty}}{|D-\lambda_m^\infty|}\frac{\| \operatorname{diag}((\eta_i(\kappa,\mathfrak{I}))_{i=1}^m) \oplus \operatorname{diag}((\eta_i(\kappa,\mathfrak{I}))_{i=1}^m) \|}{\sqrt{1-\eta_m(\kappa,\mathfrak{I})}}.$$

In the case in which $\mathfrak{I} = \big[D_-, D_+\big]$ and $\lambda_{q-1}^\kappa < D_- \leq \lambda_q^\kappa \leq \lambda_{q+m-1}^\kappa \leq D_+ < \lambda_{q+m}^\kappa$, $\kappa > \kappa_0$ then

$$\|E_\kappa(\mathfrak{I}) - E_\infty(\mathfrak{I})\| \leq \Big[\frac{\sqrt{D_+\lambda_m^\infty}}{|D_+ - \lambda_m^\infty|} + \frac{\sqrt{\lambda_1^\infty D_-}}{|\lambda_1^\infty - D_-|}\Big]\frac{\eta_m(\kappa,\mathfrak{I})}{\sqrt{1-\eta_m(\kappa,\mathfrak{I})}}. \tag{3.20}$$

An easy comparison with the single operator estimates from Theorem 3.3 reveals that, unlike the spectral family estimate (3.19), the eigenvalue result (3.18) is suboptimal in the asymptotic setting. The problem is that we can not uniformly apply the estimate (3.12) on all the operators $\mathbf{H}_\kappa$, $\kappa > \kappa_0$ since we have no information of the multiplicity of the eigenvalue λ_i^κ for all $\kappa > \kappa_0$. We only know the multiplicity of λ_i^∞. The only statement which we can make in general is a lower estimate on the convergence rate. A way to solve this multiplicity problem will be presented in Section 3.2.4, for now we only give the following result.

Corollary 3.8. *Assuming the setting and the notation of Theorem 3.5 we have*

$$\frac{\lambda_1^\infty}{2\lambda_m^\infty}\sum_{i=1}^m \eta_i^2(\kappa,\mathfrak{I}) \leq \sum_{i=1}^m \frac{|\lambda_i^\kappa-\lambda_i^\infty|}{\lambda_i^\infty}.$$

Furthermore, for each $\kappa>0$ *we can chose eigenvectors* v_i^κ, $\mathbf{H}_\kappa v_i^\kappa=\lambda_i^\kappa v_i^\kappa$, $\|v_i^\kappa\|=1$ *and* v_i^∞, $\mathbf{H}_\infty v_i^\infty=\lambda_i^\infty v_i^\infty$, $\|v_i^\infty\|=1$ *such that*

$$\frac{\lambda_1^\infty}{2\lambda_m^\infty}\sum_{i=1}^m \eta_i^2(\kappa,\mathfrak{I}) \leq \sum_{i=1}^m \frac{\mathfrak{h}_\kappa[v_i^\kappa-v_i^\infty]}{\mathfrak{h}\kappa[v_i^\infty]}.$$

One situation in which we can readily obtain upper estimates like those from Theorem 3.3 is the case when we know that λ_i^∞ has the multiplicity one. This is frequently a case for the $1D$ differential operators. Also, the lowest eigenvalue of many Schroedinger operators, like those from [7] have multiplicity one. In what follows we use $\|\cdot\|_{\mathbf{A}^{-1}}=\|\mathbf{A}^{-1/2}\cdot\|$ to denote the standard $\mathbf{A}^{-1}$-norm, which is associated to a positive definite operator $\mathbf{A}$.

Theorem 3.9. *Assume the setting and the notation of Theorem 3.5, and let* λ_q^∞, $q\in\mathbb{N}$ *be of multiplicity one then*

$$\lim_{\kappa\to\infty}\frac{\frac{\lambda_q^\infty-\lambda_q^\kappa}{\lambda_q^\infty}}{\eta_1^2(\kappa,\lambda_q^\infty)}=1, \tag{3.21}$$

$$\lim_{\kappa\to\infty}\frac{\frac{\mathfrak{h}_\kappa[v_q^\kappa-v_q^\infty]}{\mathfrak{h}_\kappa[v_q^\kappa]}}{\eta_1^2(\kappa,\lambda_q^\infty)}=1 \tag{3.22}$$

Proof. By the same argument as above we may assume that we have κ_0 such that

$$\eta_1(\kappa,\lambda_q^\infty)\leq\frac{1}{3}\min\{\frac{\lambda_{q+1}^\infty-\lambda_q^\infty}{\lambda_{q+1}^\infty+\lambda_q^\infty},\frac{\lambda_q^\infty-\lambda_{q-1}^\infty}{\lambda_q^\infty+\lambda_{q-1}^\infty}\},\quad \kappa>\kappa_0.$$

Theorem 3.5 yields that there exist D_-,D_+ such that $0<D_-<\lambda_q^\infty<D_+$ and

$$\lambda_{q-1}^\kappa<D_-<\lambda_q^\kappa<D_+<\lambda_{q+1}^\kappa,\qquad \kappa>\kappa_0. \tag{3.23}$$

According to [13] we conclude that we may apply the error representation formula (3.9) to the operator $\mathbf{H}_\kappa$ and the test vector v_q^∞, such that $\mathbf{H}_\infty v_q^\infty=\lambda_q^\infty v_q^\infty$, $\|v_q^\infty\|=1$. To the vector v_q^∞ we can define the *residuum* as the functional $\mathfrak{r}_q^\kappa:=\mathbf{H}_\kappa v_q^\infty-\lambda_q^\infty v_q^\infty$ and the identity

$$\|\mathfrak{r}_q^\kappa\|^2_{(\mathbf{H}_\kappa)^{-1}_{\mathfrak{E}_\infty}}=(v_q^\infty,\mathbf{H}_\infty v_q^\infty)\,\eta_1^2(\kappa,\lambda_q^\infty)$$

can be established by an easy computation. Also note the following identities

$$\begin{aligned}\|\mathfrak{r}_q^\kappa\|_{(\mathbf{H}_\kappa)^{-1}_{\mathfrak{E}_\infty}}&=\max_{v\in\mathcal{Q}\backslash\{0\}}\frac{|\langle\mathfrak{r}_q^\kappa,v\rangle|}{\|(\mathbf{H}_\kappa)^{1/2}_{\mathfrak{E}_\infty}v\|}=\max_{v\in\mathcal{Q}\backslash\{0\}}\frac{|\mathfrak{h}_\kappa(v,v_q^\infty)-(\mathfrak{h}_\kappa)_{\mathfrak{E}_\infty}(v,v_q^\infty)|}{\|(\mathbf{H}_\kappa)^{1/2}_{\mathfrak{E}_\infty}v\|}\\&=\max_{\substack{v\in\mathcal{Q}\backslash\{0\}\\ v\perp\mathsf{N}(\mathfrak{h}_e)}}\frac{|\mathfrak{h}_\kappa(v,v_q^\infty)-(\mathfrak{h}_\kappa)_{\mathfrak{E}_\infty}(v,v_q^\infty)|}{\|(\mathbf{H}_\kappa)^{1/2}_{\mathfrak{E}_\infty}v\|},\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the standard duality product. Analogous manipulation and the error representation formula (3.9) yield the conclusion

$$\begin{aligned}\frac{\frac{\lambda_q^\infty-\lambda_q^\kappa}{\lambda_q^\infty}}{\eta_1^2(\kappa,\lambda_q^\infty)}&=1+\frac{\lambda_q^\kappa}{\lambda_q^\infty}\frac{((\mathbf{H}_\kappa)^{-1}_{\mathfrak{E}_\infty}\mathfrak{r}_q^\kappa,(\mathbf{I}-\lambda_q^\kappa(\mathbf{H}_\kappa)^{-1}_{\mathfrak{E}_\infty})^{-1}(\mathbf{H}_\kappa)^{-1}_{\mathfrak{E}_\infty}\mathfrak{r}_q^\kappa)}{\eta_1^2(\kappa,\lambda_q^\infty)}\\&=1+O\big(\|(\mathbf{H}_\kappa)^{-1/2}_{\mathfrak{E}_\infty}P_{\mathsf{N}(\mathfrak{h}_e)_\perp}\|^2\big).\end{aligned}$$

Finally, Theorem 2.1 implies (3.21). The conclusion (3.22) follows from (3.14) □

We would like to emphasize that in this result the monotonicity of the family $\mathfrak{h}_\kappa$ played a role. It is possible to prove the result without the property $\lambda_q^\infty > \lambda_q^\kappa$. The proof is technically more involved and it does not further the understanding of the problem, so we leave it out. This theorem establishes that the estimate—which follows directly from Theorem 3.2—is sharp. We formulate this as the following corollary.

Corollary 3.10. *Assume the setting of the preceding theorem then*

$$\frac{|\lambda_q^\infty - \lambda_q^\kappa|}{\lambda_q^\infty} \leq \frac{3\ \eta_1^2(\kappa, \lambda_q^\infty)}{\min\{\frac{\lambda_{q+1}^\infty - \lambda_q^\infty}{\lambda_{q+1}^\infty + \lambda_q^\infty}, \frac{\lambda_q^\infty - \lambda_{q-1}^\infty}{\lambda_q^\infty + \lambda_{q-1}^\infty}\}}.$$

This estimate is sharp in the sense of (3.21).

3.2.1. *A concrete example.* Let $\mathbf{H}_\kappa$ be the operators which are defined by the family of positive definite forms

$$\mathfrak{h}_\kappa(u, v) = \int_0^\infty \partial_x u \partial_x v\ dx + \kappa^2 \int_1^\infty uv\ dx, \quad u, v \in H_0^1(\mathbb{R}_+). \tag{3.24}$$

Theorem 2.1 readily yields

$$\mathfrak{h}_\infty(u, v) = \int_0^1 \partial_x u \partial_x v\ dx, \qquad u, v \in H_0^1[0, 1].$$

Here we have used $H_0^1[0,1]$ and $H_0^1(\mathbb{R}_+)$, $\mathbb{R}_+ := \big[0, \infty\big\rangle$ to denote the standard Sobolev spaces. We also identify the functions from $H_0^1[0,1]$ with their extension by zero to the whole of $\mathbb{R}_+$ and write $H_0^1[0,1] \subset H_0^1(\mathbb{R}_+)$. We also formally write $\mathbf{H}_\kappa = -\partial_{xx} + \kappa^2 \chi_{[1,\infty\rangle}$ and $\mathbf{H}_\infty = -\partial_{xx}$ and chose

$$u_i(x) = \begin{cases} \sqrt{2}\sin(k\pi x), & 0 \leq x \leq 1 \\ 0, & 1 \leq x \end{cases}, i \in \mathbb{N} \tag{3.25}$$

as a test function(s). A simple computation yields that λ_i^κ is a solution of the equation

$$\sqrt{\kappa^2 - \lambda^\kappa} = -\sqrt{\lambda^\kappa} \cot(\sqrt{\lambda^\kappa}) \tag{3.26}$$

and we know that each λ_i^κ has the multiplicity one. The quotient $\frac{\lambda_1^\infty - \lambda_1^\kappa}{\lambda_1^\infty}$ can be represented (for $\kappa \to \infty$) by a convergent Taylor series (see [25])

$$\frac{\lambda_1^\infty - \lambda_1^\kappa}{\lambda_1^\infty} = 2\frac{1}{\kappa} - 3\frac{1}{\kappa^2} + 8\left(\frac{1}{2!} + \frac{1}{4!}\pi^2\right)\frac{1}{\kappa^3} - 10\left(\frac{1}{2!} + \frac{4}{4!}\pi^2\right)\frac{1}{\kappa^4} + \cdots . \tag{3.27}$$

Using the Green functions we also directly compute $\eta_1^2(\kappa, \lambda_i^\infty) := \frac{2}{3+\kappa}$. For computational details see [14].

By utilizing the information from (3.26) we can establish

$$\big(1 - \sqrt{\frac{2}{3+\kappa}}\big)4\pi^2 =: D(\kappa) \leq \lambda_2(\mathbf{H}), \qquad \kappa \geq 5, \tag{3.28}$$

which leads, in combination with (3.12), to the estimate

$$\frac{2}{3+\kappa} \leq \frac{\lambda_1^\infty - \lambda_1^\kappa}{\lambda_1^\infty} \leq \frac{D(\kappa) + \pi^2}{D(\kappa) - \pi^2}\frac{2}{3+\kappa} = \frac{10}{3\kappa} + \frac{1}{\sqrt{\kappa}}O\Big(\frac{1}{\kappa}\Big), \qquad \kappa \geq 5. \tag{3.29}$$

Similar sharp results can be obtained for other λ_i^∞ and using (3.14) for corresponding eigenvectors. We tacitly leave out the details.

3.2.2. *A remark on higher dimensional singular obstacle problems.* This paradigm has been applied in [14] to operators which are defined both in $H^1(\mathbb{R}^n)$ as well as in $H^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. The only ingredient which is necessary is a result on the behavior of the momenta

$$(f, \mathbf{H}_\kappa^{-1} f) - (f, \mathbf{H}_\infty^\dagger f) = \int_{\kappa^2}^\infty \|\mathbf{H}_e^{1/2} \mathbf{H}_\tau^{-1} f\|^2 \, d\tau, \qquad f \in \mathfrak{E}_\infty. \tag{3.30}$$

Estimates of such momenta have been obtained on many places in the literature. We illustrate our point by a consideration of a model problem of the electro-magnetic waveguide $\mathcal{O} \times \mathbb{R}$, where the section $\mathcal{O} \subset \mathbb{R}^2$ is a smooth and connected domain. The material $\Omega \subset \mathcal{O}$ of very large conductivity is compactly immersed in $\mathcal{O}$, which is to say that the closure cl(Ω) is contained in $\mathcal{O}$ and that Ω is bounded. The dielectric material is now modeled by $\mathcal{U} = \mathcal{O} \setminus \Omega$. Assuming that the boundary of Ω is sufficiently smooth we study the eigenvalue problem for $\mathbf{H}_\kappa = -\triangle + \kappa^2 \chi_\Omega$. Here, χ_Ω is the characteristic function of Ω and $\mathbf{H}_\kappa$ is the operator which is defined in the sense of Kato by the quadratic form

$$\mathfrak{h}_\kappa(\psi, \phi) = \int_{\mathcal{O}} \nabla\psi \cdot \nabla\phi + \kappa^2 \int_{\mathcal{O}} \chi_\Omega \psi\phi, \qquad \psi, \phi \in \mathcal{Q}_\infty := H_0^1(\mathcal{O})$$

where $\kappa \in \mathbb{R}_+$ and ∇ is the usual gradient operator on $H_0^1(\mathcal{O})$, the Sobolev space of functions with zero trace on the boundary $\partial\mathcal{O}$.

This problem has been analyzed in [12]. Let us assume that $\lambda_m^\infty < D < \lambda_{m+1}^\infty$, for some $m \in \mathbb{N}$. We compare $P = E_\infty(\lambda_m^\infty)$ and $Q_\kappa = E_\kappa(\lambda_m^\infty)$, where $\mathbf{H}_\kappa = \int \lambda \, dE_\kappa(\lambda)$ is the spectral integral in $L^2(\mathcal{O})$ and $\mathbf{H}_\infty = \int \lambda \, dE_\infty(\lambda)$ is the spectral integral in $L^2(\mathcal{U})$. Since, as has been shown in [12],

$$(v_i^\infty, \mathbf{H}_\kappa^{-1} v_i^\infty) - (v_i^\infty, \mathbf{H}_\infty^\dagger v_i^\infty) = \frac{1}{\kappa} \frac{1}{(\lambda_i^\infty)^2} \int_{\partial\Omega} \frac{\partial v_i^\infty}{\partial\nu} \frac{\partial v_i^\infty}{\partial\nu} + O\big(\frac{1}{\kappa^{3/2}}\big),$$

we have coarse eigenvector estimates

$$\|Q_\kappa - P\| \le \frac{\sqrt{D\lambda_m^\infty}}{D - \lambda_m^\infty} \frac{1}{\sqrt{\kappa}} + O\big(\frac{1}{\kappa^{3/4}}\big) \le \frac{4}{\frac{\lambda_{m+1}^\infty - \lambda_m^\infty}{\lambda_{m+1}^\infty + \lambda_m^\infty}} \frac{1}{\sqrt{\kappa}}, \tag{3.31}$$

$$\min_{i=1,\cdots,m} \frac{\int_{\partial\Omega} \frac{\partial v_i^\infty}{\partial\nu} \frac{\partial v_i^\infty}{\partial\nu}}{\lambda_i^\infty} \frac{1}{2\kappa} \le \frac{\mathfrak{h}_\kappa[v_i^\kappa - v_i^\infty]}{\mathfrak{h}_\kappa[v_i^\infty]} \le \frac{4}{\frac{\lambda_{m+1}^\infty - \lambda_m^\infty}{\lambda_{m+1}^\infty + \lambda_m^\infty}} \frac{1}{\kappa} \tag{3.32}$$

which can be improved in a straight forward manner by bringing the factor $\int_{\partial\Omega} \frac{\partial v_i^\infty}{\partial\nu} \frac{\partial v_i^\infty}{\partial\nu}$ into estimates, as has been shown in [12, Section 2.1]. The last inequality in (3.31) and (3.32) hold for κ large enough. The optimal eigenvalue estimate can easily be constructed from Theorem 3.3 and 3.9 and we know that the eigenvector estimate (3.32) is optimal in the sense of (3.14) and (3.22). Remark 3.6 indicates how to assess the radius of convergence of these first order estimate(s).

3.2.3. *Remarks on (finite) eigenvalues in gaps of essential spectrum and on general converging families* $\mathfrak{h}_\kappa$. We have said that the theory can be applied to eigenvalues which are in the gaps of essential spectrum. Since we do not consider any model examples which show such behavior (e.g. operators with periodic boundary conditions) we will only briefly outline a possibility to obtain results like Theorem 3.9 or Theorem 3.5 in this setting.

In dealing with the eigenvalues in gaps of the essential spectrum we do not have the safe convergence environment of Theorem 2.1. Instead, we have to have an *a priori* information that the assumption like (3.23) holds. An example of how to obtain this type of *a*

priori information can be seen on the proof of (3.28). To this end we would like to emphasize that such type of "precise" result on the separation of the target eigenvalue from the unwanted component of the spectrum is an unavoidable ingredient of all constructive spectral estimates. An assumption like (3.23) is equivalent to requiring that eigenvalue λ_q be stable under the perturbation $\mathfrak{h}_\kappa$, see [20, chapter VIII.4, pp. 437]. For a characterization of perturbations for which this holds see [23] and references therein.

Given such an estimate—i.e. assuming that λ_q is a stable eigenvalue—the appropriate result from [13] or [17] can be applied to obtain convergence estimates. We also emphasize that the theory of [13, 17] allows for more general spectral intervals $\mathfrak{I}$. To be more precise, to establish an estimate like (3.20) the spectrum in $\mathfrak{I}\cap\mathrm{spec}(\mathbf{H}_\infty)$ does not have to be discrete. However, in such situation we have no guarantee that $\eta_{\max}(\kappa,\mathfrak{I})<1$ and obtaining computational formulae requires much more technical work. The precise use in a given situation is application dependent, but always follows the procedure outlined in Theorems 3.5 and 3.9.

In the case in which we consider a general converging family of quadratic forms from [35] we cannot conclude that $(\mathbf{H}_\infty^\dagger)_{E_\infty(\mathfrak{I})} = (\mathbf{H}_\kappa)^{-1}_{E_\infty(\mathfrak{I})}$, so we have to use explicitly computable $\Xi_\kappa^{-1} := (\mathbf{H}_\kappa)^{-1}_{E_\infty(\mathfrak{I})}\big|_{E_\infty(\mathfrak{I})}$ in (3.16) instead. If we set $\mu_i^\kappa := \lambda_i(\Xi_\kappa)$, then μ_i^κ substitutes for λ_i^∞ in eigenvalue estimates like (3.18), (3.21), whereas the estimates for the spectral projections like (3.20) remain unchanged, e.g. we have the convergence estimate $\|Q_\kappa - P\| \le \frac{\sqrt{\lambda_{m+1}^\kappa \mu_m^\kappa}}{|\lambda_{m+1}^\kappa - \mu_m^\kappa|}\frac{\eta_m(\kappa,P)}{\sqrt{1-\eta_m(\kappa,P)}}$.

3.2.4. *A method to solve the multiplicity problem.* A tacit assumption in this semiclassical analysis is that the operator $\mathbf{H}_\infty$ is a well known object. In order to be able to apply Theorem 3.3 one should establish that there exists $\kappa_0>0$ such that

$$\lambda_{q-1}^\kappa < D_- < \lambda_q^\kappa = \lambda_{q+m-1}^\kappa < D_+ < \lambda_{q+m}^\kappa, \tag{3.33}$$

for $\kappa>\kappa_0$. However, if $m>1$ it is not plausible to expect that (3.33) will hold in general. Instead, we will get a tight cluster of m eigenvalues (counting the eigenvalues according to their multiplicity) that converge to λ_q^∞. Since we aim to express the spectral information about $\mathbf{H}_\kappa$ in terms of the spectrum of $\mathbf{H}_\infty$ we further opt to give specific values for D_- and D_+ as functions of the gaps in the spectrum of $\mathbf{H}_\infty$.

Theorem 3.11. *Let the eigenvalues of the operator $\mathbf{H}_\infty$ be so ordered that $\lambda_{q-1}^\infty < \lambda_q^\infty = \lambda_{q+m-1}^\infty < \lambda_{q+m}^\infty$. Define the measure of the relative separation of λ_q^∞ from the rest of the spectrum of $\mathbf{H}_\infty$ as the number*

$$\gamma_s(\lambda_q^\infty) = \min\left\{\frac{\lambda_{q+m}^\infty - \lambda_q^\infty}{\lambda_{q+m}^\infty + \lambda_q^\infty}, \frac{\lambda_q^\infty - \lambda_{q-1}^\infty}{\lambda_q^\infty + \lambda_{q-1}^\infty}\right\}.$$

There exists $\kappa_0>0$ such that for $\kappa\ge\kappa_0$

$$\frac{|\lambda_{q+i-1}^\kappa - \lambda_q^\infty|}{\lambda_q^\infty} < \eta_m(\kappa,\lambda_q^\infty)\frac{\frac{3\eta_m(\kappa,\lambda_q^\infty)}{\gamma_c(\lambda_m^\infty)}}{1-\frac{3\eta_m(\kappa,\lambda_q^\infty)}{\gamma_s(\lambda_m^\infty)}}, \quad i=1,\dots,m. \tag{3.34}$$

Proof. Since $\eta_m(\kappa, \lambda_q^\infty) \to 0$, an argument analogous to the argument that led to Theorem 3.5 implies that we can pick $\kappa_0 > 0$ such that for $\kappa > \kappa_0$

$$\eta_m(\kappa, \lambda_q^\infty) \leq \frac{1}{3}\gamma_s(\lambda_q^\infty) \tag{3.35}$$

$$|\lambda_k^\kappa - \lambda_q^\infty| \leq \frac{1}{3}\gamma_s(\lambda_q^\infty)\lambda_q^\infty, \qquad k = q, q+1, \dots, q+m-1, \tag{3.36}$$

$$|\zeta - \lambda_k(\widehat{\mathbf{H}}_\kappa)| > \frac{1}{3}\gamma_s(\lambda_q^\infty)\lambda_k(\widehat{\mathbf{H}}_\kappa), \;\; k \notin \{q, \dots, q+m-1\}, \;\; \zeta \in \mathfrak{C}(\lambda_q^\infty). \tag{3.37}$$

Here $\mathfrak{C}(\lambda_q^\infty)$ is the circle in the complex plane with the radius $\frac{1}{3}\gamma_s(\lambda_q^\infty)\lambda_q^\infty$ and the center λ_q^∞. Assume $\kappa > \kappa_0$ is fixed, then define the family

$$\mathfrak{a}(\tau) = (\mathfrak{h}_\kappa)_P + \tau\delta_P(\mathfrak{h}_\kappa), \qquad \tau \in \mathbb{C}. \tag{3.38}$$

This is a *holomorphic family of type (B)* (for the definition see [20, Chapter VII]). We know that

$$|\delta_P(\mathfrak{h}_\kappa)[u]| < \eta_m(\kappa, \lambda_q^\infty)(\mathfrak{h}_\kappa)_P[u], \qquad u \in \mathcal{Q}, \tag{3.39}$$

so [20, Theorem VII-4.9 and (VII-4.45)] imply that the resolvent

$$R(\tau, \zeta) = (\mathbf{A}(\tau) - \zeta\mathbf{I})^{-1}$$

can be represented by a convergent power series in τ for $\zeta \in \mathfrak{C}(\lambda_q^\infty)$. The power series for $R(\tau, \zeta)$ converges for every

$$|\tau| < r_0 = \frac{1}{\eta_m(\kappa, \lambda_q^\infty)} \inf_{\substack{\zeta \in \mathfrak{C}(\lambda_q^\infty), \\ \lambda \in \operatorname{spec}((\mathbf{H}_\kappa)_P)}} \frac{|\lambda - \zeta|}{\lambda} = \frac{1}{\eta_m(\kappa, \lambda_q^\infty)} \frac{1}{3}\gamma_s(\lambda_q^\infty). \tag{3.40}$$

In particular, assumption (3.35) implies that the series converges for $\tau = 1$.

Define

$$\widehat{B}(\tau) := -\frac{1}{2\pi i}\mathbf{A}(\tau) \int_{\mathfrak{C}(\lambda_q^\infty)} R(\tau, \zeta)\, d\zeta,$$

then $\widehat{B}(\tau)$ is a holomorphic operator family and there exist m holomorfic functions $\widehat{\lambda}_i(\tau)$ such that $\widehat{\lambda}_1(\tau), \cdots, \widehat{\lambda}_m(\tau)$ are all the nonzero eigenvalues of the operator $\widehat{B}(\tau)$. Due to the assumptions we have made it follows that for $i = 1, \dots, m$

$$|\widehat{\lambda}_i(\tau) - \lambda_q^\infty| < \frac{1}{3}\gamma_s(\lambda_q^\infty)\lambda_q^\infty, \qquad |\tau| < r_0.$$

Cauchy's integral inequality[4] for the coefficients of the Taylor expansion implies, for every $i = 1, \dots, m$, the estimate

$$|\widehat{\lambda}_i^{(n)}| < \frac{\frac{1}{3}\gamma_s(\lambda_q^\infty)\lambda_q^\infty}{r_0^n}, \qquad n = 1, 2, \cdots$$

where $\widehat{\lambda}_i(\tau) = \lambda_q^\infty + \tau\widehat{\lambda}_i^{(1)} + \tau^2\widehat{\lambda}_i^{(2)} + \tau^3\widehat{\lambda}_i^{(3)} + \cdots$. This yields

$$|\widehat{\lambda}_i(\tau) - \lambda_q^\infty - \tau\widehat{\lambda}_i^{(1)}| < \frac{\frac{1}{3}\gamma_s(\lambda_q^\infty)\lambda_q^\infty}{r_0} \frac{|\tau|^2}{r_0 - |\tau|} \leq \frac{\frac{1}{3}\gamma_s(\lambda_q^\infty)\lambda_q^\infty}{r_0^2} \frac{|\tau|^2}{1 - \frac{|\tau|}{r_0}}$$

[4]For further details see [1, Section 8.1.4] and [20, Section II-3].

for $|\tau| < r_0$. In particular for $\tau = 1$ there exists a permutation $j_{(\cdot)}$ such that $\widehat{\lambda}_{j_i}(1) = \lambda^\kappa_{q+i-1}$, $i = 1, \dots, m$ so

$$|\lambda^\kappa_{q+i-1} - \lambda^\infty_q - \widehat{\lambda}^{(1)}_{j_i}| < \eta_m(\kappa, \lambda^\infty_q)\lambda^\infty_q \; \frac{3\eta_m(\kappa, \lambda^\infty_q)}{\gamma_c(\lambda^\infty_q)} \; \frac{1}{1 - \frac{3\eta_m(\kappa,\lambda^\infty_q)}{\gamma_s(\lambda^\infty_q)}}.$$

With this is the proof of the theorem finished. To see this note that it was established, in [20, (VII-4.50)], that $\widehat{\lambda}^{(1)}_{j_i}$ are the eigenvalues of the matrix $M_{kp} = \delta_P(\mathfrak{h}_\kappa)(u_k, u_p)$, where u_k, $k = 1, \dots, m$ form an orthonormal basis for $\mathsf{R}(E_\infty[D_-D_+])$. Since

$$\delta_P(\mathfrak{h}_\kappa)(u, v) = \mathfrak{h}_\kappa(P_\perp u, Pu) + \mathfrak{h}_\kappa(Pu, P_\perp u) = 0, \qquad u, v \in \mathsf{R}(P),$$

we obtain $\widehat{\lambda}^{(1)}_{j_i} = 0$, $i = 1, \dots, m$ and the conclusion follows. $\square$

Remark 3.12. The estimate of this theorem is optimal in the sense of Corollary 3.8. The upper estimate which has a similar form to (3.12) can be established for the limit eigenvalues $\lambda^\infty_1 \leq \cdots \leq \lambda^\infty_m$. The the role of the constant from (3.12) is taken by the constant $\gamma_{\min}(\lambda^\infty_m) := \min\{\gamma_c(\lambda^\infty_i) \; : \; i = 1, \dots, m\}$, as is given by the repeated application of Theorem 3.11. We leave out the technical details.

4. Spectral asymptotic in the regular case

We now concentrate on the non-inhibited families

$$\mathfrak{h}_\kappa(u, v) = \mathfrak{h}_b(u, v) + \kappa^2\mathfrak{h}_e(u, v), \qquad u, v \in \mathcal{Q} := \mathcal{Q}(\mathfrak{h}_b) \subset \mathcal{Q}(\mathfrak{h}_e), \tag{4.1}$$

which satisfy the additional regularity assumption that the range of the operator $\mathbf{H}_e^{1/2}\mathbf{H}_b^{-1/2}$ is closed in $\mathcal{H}$. As already mentioned in Section 1.2 this is equivalent with

$$\|(\mathbf{H}_e^{1/2}\mathbf{H}_b^{-1/2})^\dagger\| = \mathfrak{k} < \infty. \tag{4.2}$$

With this additional requirement, which has a flavor of Linear Algebra, we can use an adaptation of the Lagrange-Multiplier technique to establish an upper estimate for the momenta

$$(f, \mathbf{H}_\kappa^{-1} f) - (f, \mathbf{H}_\infty^\dagger f) = \int_{\kappa^2}^\infty \|\mathbf{H}_e^{1/2}\mathbf{H}_\tau^{-1} f\|^2 \, d\tau, \qquad f \in \mathcal{Q}_\infty := \mathcal{Q}(\mathfrak{h}_\infty). \tag{4.3}$$

The lower estimate for (4.3) follows by an adaptation of the spectral-calculus technique from [3, 4]. With this we prove the optimality of our approach to spectral asymptotic estimation.

The following lemmata are the main technical results which are needed to estimate the quantities (4.3).

Lemma 4.1. *Take $f \in \overline{\mathcal{Q}_\infty}^{\,\|\cdot\|}$, then $\mathfrak{h}_\kappa[\mathbf{H}_\kappa^{-1} f - \mathbf{H}_\infty^\dagger f] = (f, \mathbf{H}_\kappa^{-1} f) - (f, \mathbf{H}_\infty^\dagger f)$.*

Proof. The proof is a straight forward computation. Take $f \in \overline{\mathcal{Q}_\infty}^{\,\|\cdot\|}$, then

$$h_\kappa[\mathbf{H}_\infty^\dagger f] = (f, \mathbf{H}_\infty^\dagger f)$$

and we have

$$\begin{aligned} h_\kappa[\mathbf{H}_\kappa^{-1} f - \mathbf{H}_\infty^\dagger f] &= (f, \mathbf{H}_\kappa^{-1} f) - h_\kappa(\mathbf{H}_\kappa^{-1} f, \mathbf{H}_\infty^\dagger f) - h_\kappa(\mathbf{H}_\infty^\dagger f, \mathbf{H}_\kappa^{-1} f) + (f, \mathbf{H}_\infty^\dagger f) \\ &= (f, \mathbf{H}_\kappa^{-1} f) - (\mathbf{H}_\kappa^{-1/2} f, \mathbf{H}_\kappa^{1/2}\mathbf{H}_\infty^\dagger f) - (\mathbf{H}_\kappa^{1/2}\mathbf{H}_\infty^\dagger f, \mathbf{H}_\kappa^{-1/2} f) \\ &\quad + (f, \mathbf{H}_\infty^\dagger f) \\ &= (f, \mathbf{H}_\kappa^{-1} f) - (f, \mathbf{H}_\infty^\dagger f). \end{aligned}$$

$\square$

Lemma 4.2. *Let $f \in \mathcal{H}$ be given then set $r_f := \mathbf{H}_b^{-1/2} f - \mathbf{H}_b^{1/2}\mathbf{H}_\infty^\dagger f$. If we assume $\|(\mathbf{H}_e^{1/2}\mathbf{H}_b^{-1/2})^\dagger\| < \infty$ then $q_f = (\mathbf{H}_e^{1/2}\mathbf{H}_b^{-1/2})^\dagger r_f$ and*

$$\mathfrak{h}_b(\mathbf{H}_\infty^\dagger f, v) + (q_f, \mathbf{H}_e^{1/2} v) = (f, v), \qquad v \in \mathcal{Q}.$$

Furthermore, it holds that $\|r_f\|^2 = (f, \mathbf{H}_b^{-1} f) - (f, \mathbf{H}_\infty^\dagger f)$.

Proof. It holds that $r_f \perp \mathbf{H}_b^{1/2}\mathcal{Q}_\infty$, which can be checked by a direct computation. The operator $\mathbf{B} := (\mathbf{H}_e^{1/2}\mathbf{H}_b^{-1/2})$ has the closed range so

$$\mathcal{H} = \mathsf{R}(\mathbf{B}^*) \oplus \mathsf{N}(\mathbf{B}) = \mathsf{R}(\mathbf{B}^*) \oplus \mathbf{H}_b^{1/2}\mathcal{Q}_\infty.$$

Therefore we have $r_f \in \mathsf{R}(\mathbf{B}^*)$ and so we may write $q_f := \mathbf{B}^\dagger r_f$. A direct computation now shows that

$$\begin{aligned}\mathfrak{h}_b(\mathbf{H}_\infty^\dagger f, v) + (q_f, \mathbf{H}_e^{1/2} v) &= (\mathbf{H}_b^{1/2}\mathbf{H}_\infty^\dagger f, \mathbf{H}_b^{1/2} v) + (\mathbf{B}^{*\dagger} r_f, \mathbf{B}\mathbf{H}_b^{1/2} v)\\ &= (\mathbf{H}_b^{1/2}\mathbf{H}_\infty^\dagger f + r_f, \mathbf{H}_b^{1/2} v) = (f, v).\end{aligned}$$

□

The main quantitative theorem about the asymptotic behavior of (4.1) follows now directly.

Theorem 4.3. *Assume $\mathfrak{k} := \|(\mathbf{H}_e^{1/2}\mathbf{H}_b^{-1/2})^\dagger\| < \infty$ then we have; for $f \in \overline{\mathcal{Q}_\infty}^{\|\cdot\|}$;*

$$\frac{(f,\mathbf{H}_1^{-1}f) - (f,\mathbf{H}_\infty^\dagger f)}{\kappa^2} \leq (f, \mathbf{H}_\kappa^{-1} f) - (f, \mathbf{H}_\infty^\dagger f) \leq \frac{\mathfrak{k}^2\big((f,\mathbf{H}_b^{-1}f) - (f,\mathbf{H}_\infty^\dagger f)\big)}{\kappa^2} \tag{4.4}$$

and

$$\frac{1}{\kappa^2}\eta_i^2(1,\lambda^\infty) \leq \eta_i^2(\kappa,\lambda^\infty) \leq \frac{\mathfrak{k}^2}{\kappa^2}\eta_i^2(0,\lambda^\infty), \qquad i = 1,\ldots,m\,, \tag{4.5}$$

where m is the multiplicity of the discrete eigenvalue λ^∞ (not necessarily below the infimum of the essential spectrum of $\mathbf{H}_\infty$).

Proof. For any $f \in \mathcal{H}$ we have

$$\begin{aligned}\mathfrak{h}_b(\mathbf{H}_\infty^\dagger f, v) + (q_f, \mathbf{H}_e^{1/2} v) &= (f, v), \qquad v \in \mathcal{Q},\\ \mathfrak{h}_b(\mathbf{H}_\kappa^{-1} f, v) + \kappa^2\mathfrak{h}_e(\mathbf{H}_\kappa^{-1} f, v) &= (f, v), \qquad v \in \mathcal{Q}.\end{aligned}$$

which implies

$$\mathfrak{h}_b(\mathbf{H}_\kappa^{-1} f - \mathbf{H}_\infty^\dagger f, v) + \kappa^2\mathfrak{h}_e(\mathbf{H}_\kappa^{-1} f, v) = (q_f, \mathbf{H}_e^{1/2} v)$$

and subsequently

$$\kappa^2\mathfrak{h}_e[\mathbf{H}_\kappa^{-1} f] \leq \|q_f\|\mathfrak{h}_e[\mathbf{H}_\kappa^{-1} f]^{1/2}.$$

The right inequality in (4.4) follows from Lemma 4.2. To establish the left inequality of (4.4) we start from the identity [3, (22)]. We combine the integral representation for $(f, \mathbf{H}_\kappa^{-1} f) - (f, \mathbf{H}_\infty^\dagger f)$ from [3, pp. 41] and [3, (29)] to obtain

$$\begin{aligned}(f, \mathbf{H}_\kappa^{-1} f) - (f, \mathbf{H}_\infty^\dagger f) &= \int_0^\infty \frac{1}{\lambda + \kappa^2\lambda^2}(dE_{\mathbf{H}_e}(\lambda)\mathbf{H}_e^{1/2}\mathbf{H}_b^{-1} f, \mathbf{H}_e^{1/2}\mathbf{H}_b^{-1} f)\\ &= ((\mathbf{I} + \kappa^2\mathbf{H}_e)^{-1}\mathbf{H}_b^{-1} f, \mathbf{H}_b^{-1} f)\\ &\geq \frac{1}{\kappa^2}((\mathbf{I} + \mathbf{H}_e)^{-1}\mathbf{H}_b^{-1} f, \mathbf{H}_b^{-1} f)\\ &= \frac{1}{\kappa^2}\big((f, \mathbf{H}_1^{-1} f) - (f, \mathbf{H}_\infty^\dagger f)\big).\end{aligned}$$

The conclusion (4.5) for the approximation defects follows directly from the definition (3.16) and the observaton that

$$\frac{1}{(f,\mathbf{H}_1^{-1}f)} \leq \frac{1}{(f,\mathbf{H}_\kappa^{-1}f)} \leq \frac{1}{(f,\mathbf{H}_\infty^{\dagger}f)}, \qquad f \in \mathsf{R}\big(E_\infty(\{\lambda^q\})\big),$$

holds. This completes the argument. □

Example 4.4. We will present this example as an abstract variation on (1.2). Let $\mathbf{H}$ be a positive definite operator, let P be a projection, $\mathsf{R}(P) \subset \mathcal{D}(\mathbf{H}^{1/2})$ and let $r_f^\kappa := \mathbf{H}_\kappa^{-1/2}f - \mathbf{H}_\kappa^{1/2}\mathbf{H}_\infty^\dagger f$. Consider

$$\mathfrak{h}_\kappa(u,v) = ((\mathbf{I}+\kappa^2 P)\mathbf{H}^{1/2}u, \mathbf{H}^{1/2}v) = \mathfrak{h}_b(u,v) + \kappa^2\mathfrak{h}_e(u,v),$$

then

$$\|(\mathbf{H}_e^{1/2}\mathbf{H}_b^{-1/2})^\dagger\| \leq 1$$

and (4.4) gives for $f \in \mathsf{N}(P\mathbf{H}_e^{1/2})$

$$\frac{\|r_f^\kappa\|^2}{\|r_f\|^2} = \frac{(f,\mathbf{H}_\kappa^{-1}f) - (f,\mathbf{H}^{-1/2}P_\perp\mathbf{H}^{-1/2}f)}{(\mathbf{H}^{-1/2}f, P\mathbf{H}^{-1/2}f)} \leq \frac{1}{\kappa^2}. \tag{4.6}$$

Here we have used $\|r_f\|^2 = (f,\mathbf{H}_b^{-1}f) - (f,\mathbf{H}_\infty^\dagger f)$ and $\|r_f^\kappa\|^2 = (f,\mathbf{H}_\kappa^{-1}f) - (f,\mathbf{H}_\infty^\dagger f)$ to simplify the notation. On the other hand, we compute

$$\mathbf{H}_\kappa^{-1} = \mathbf{H}^{-1/2}(P_\perp + \frac{1}{1+\kappa^2}P)\mathbf{H}^{-1/2}$$

to establish

$$(f,\mathbf{H}_\kappa^{-1}f) - (f,\mathbf{H}^{-1/2}P_\perp\mathbf{H}^{-1/2}f) = \frac{1}{1+\kappa^2}(\mathbf{H}^{-1/2}f, P\mathbf{H}^{-1/2}f). \tag{4.7}$$

Formulae (4.6) and (4.7) give

$$\frac{1}{1+\kappa^2} = \frac{\|r_f^\kappa\|^2}{\|r_f\|^2} \leq \frac{1}{\kappa^2},$$

which is a very favorable estimate for κ large. The lower estimate can be computed to be

$$\frac{1}{2\kappa^2} \leq \frac{\|r_f^\kappa\|^2}{\|r_f\|^2},$$

which is not as sharp as the upper estimate, but it is—newer the less—asymptotically optimal.

5. A MODEL PROBLEM FROM 1D THEORY OF ELASTICITY

As an illustration of the applicability of Theorem 4.3, we consider the small frequency problem for the circular arch as described in [6, Chapter 8.8:3] and [28], cf. Figure 1. Let $\phi : [0,l] \to \mathbb{R}^2$ be the middle curve of the arch. We take ϕ to be the upper part of the circle with the radius R. The arch (the model problem we are considering) will be a thin homogeneous, elastic body of the constant cross-section $\mathcal{A}$, whose area is $A > 0$. The arch will be clamped at one end and free at the other. The strain energy of the arch is given[5] by

[5] See also [34].

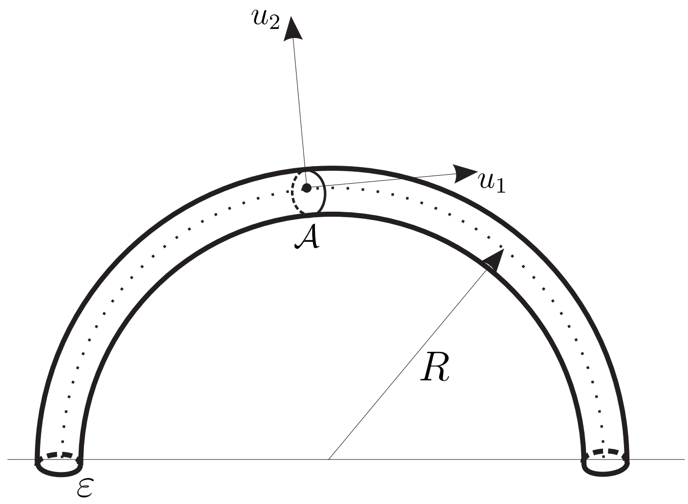

FIGURE 1. The Curved rod model

the positive definite form

(5.1)
$$\mathfrak{a}(\mathbf{u},\mathbf{v}) = EI\int_0^l \left(u_2' + \frac{u_1}{R}\right)' \left(v_2' + \frac{v_1}{R}\right)' \, ds + EA\int_0^l \left(u_1' - \frac{u_2}{R}\right)\left(v_1' - \frac{v_2}{R}\right)\, ds,$$
$$\mathbf{u},\mathbf{v} \in \mathcal{Q}(a) = \{\mathbf{u} \in H^1[0,l] \times H^2[0,l] \ : \ \mathbf{v}(0) = 0, v_2'(0) = 0\}.$$

Here $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are the functions of the curvilinear abscissa $s \in [0, l]$, the constant E is the Young modulus of elasticity, the constant A is the area of the cross-section $\mathcal{A}$ and the constant I is the moment of inertia of the cross-section $\mathcal{A}$.

Let us assume we have the referent arch with the cross-section area A and the cross-section moment I. We consider the family of rods whose cross-section and the moment of inertia of the cross-section behave like
$$A_\kappa = \frac{1}{\kappa^2} A = \varepsilon^2 A, \qquad I_\kappa = \frac{1}{\kappa^4} I = \varepsilon^4 I.$$
We want to study the spectral properties of this family of arches as $\varepsilon \to 0$. More general arch models can be treated by analogous procedures. This is a subject for future reports.

For some given $\kappa > 0$, $\kappa := \varepsilon^{-1}$, we write
$$\mathfrak{a}_\kappa(\mathbf{u},\mathbf{v}) = \frac{E\ I}{\kappa^4}\int_0^l \Big(u_2' + \frac{u_1}{R}\Big)'\Big(v_2' + \frac{v_1}{R}\Big)' \mathrm{d}s + \frac{E\ A}{\kappa^2}\int_0^l \Big(u_1' - \frac{u_2}{R}\Big)\Big(v_1' - \frac{v_2}{R}\Big)\, ds$$
and use $\mathbf{A}_\kappa$ to denote the operator which is defined by $\mathfrak{a}_\kappa$. Since $\mathbf{A}_\kappa$ has only the discrete spectrum we write $\lambda_i(\mathbf{A}_\kappa)$, $i \in \mathbb{N}$. After rescaling
$$\lambda_i(\mathbf{A}_\kappa) = \frac{1}{\kappa^4}\lambda_i^\kappa$$
we see that λ_i^κ are the eigenvalues of the operator $\mathbf{H}_\kappa$, which is defined by
$$\begin{aligned}\mathfrak{h}_\kappa(\mathbf{u},\mathbf{v}) &= \mathfrak{h}_b(\mathbf{u},\mathbf{v}) + \kappa^2\mathfrak{h}_e(\mathbf{u},\mathbf{v})\\ &= EI\int_0^l \left(u_2' + \frac{u_1}{R}\right)'\left(v_2' + \frac{v_1}{R}\right)'\, ds + \kappa^2 EA\int_0^l \left(u_1' - \frac{u_2}{R}\right)\left(v_1' - \frac{v_2}{R}\right)\, ds\end{aligned}$$

for $\mathbf{u},\mathbf{v}\in\mathcal{Q}(\mathfrak{a}_\kappa)=\mathcal{Q}(\mathfrak{h}_\kappa)$. Since λ_i^κ enable us to describe only the eigenvalues of $\mathbf{A}_\kappa$ for which

$$\lim_{\kappa\to\infty}\frac{1}{\kappa^4}\lambda_i(\mathbf{A}_\kappa)<\infty.$$

here we see where the name "*low frequency problem*", for the eigenvalue problem for $\mathbf{H}_\kappa$, comes from. The low frequency problem satisfies the conditions of Theorem 2.1, so we conclude that the limiting form is

(5.2)

$$\mathfrak{h}_\infty(\mathbf{u},\mathbf{v})=EI\int_0^l\left(u_2'+\frac{u_1}{R}\right)'\left(v_2'+\frac{v_1}{R}\right)'\,\mathrm{d}s,\quad \mathbf{u},\mathbf{v}\in\{\mathbf{f}\in\mathcal{Q}(a_\kappa),f_1'-\frac{f_2}{R}=0\}.$$

In [34] it has shown that (5.2) is the strain energy of the Curved Rod Model and that $\mathfrak{h}_\kappa$, $\kappa>0$ are positive definite with

$$\mathcal{Q}(h_\kappa)=\{\mathbf{u}\in H^1[0,l]\times H^2[0,l]\;:\;\mathbf{v}(0)=0,v_2'(0)=0\}.$$

Remark 5.1. From (5.2) we can see the significance of the condition

$$f_1'-\frac{f_2}{R}=0. \tag{5.3}$$

Assume the rod is locally straight. That is to say, assume $R\to\infty$, then (5.3) turns into

$$f_1'=0,$$

a condition of the inextensibility of the middle curve of the straight rod. The fact that $f_1'-\frac{f_2}{R}=0$ is an *inextensibility condition* for the middle curve of the curved rod can be established by a rigorous differential geometric argument, see [34]. Continuing this heuristic reasoning, we conclude that Curved Rod model describes the transversal vibrations (perpendicular to the middle curve) of the curved rod. Arch Model "couples" the longitudinal vibrations of the rod with the transversal vibrations. The study of finer properties of longitudinal vibrations requires the analysis of the so called "*middle frequency problem*", which will not be further considered here. However, since the "*middle frequency problem*" also falls under the scope of Theorem 2.1 this theory could also be applied in that case, too.

5.1. **Computational details.** Based on (5.1) and (5.2) one concludes that the sequence $\mathfrak{h}_\kappa$ satisfies the assumptions of Theorem 4.3. Here is a word of additional explanation in order. We have formulated all of our results about the forms $\mathfrak{h}_b$ and $\mathfrak{h}_e$ based on the representations

$$\mathfrak{h}_b(u,v)=(\mathbf{H}_b^{1/2}u,\mathbf{H}_b^{1/2}v),$$
$$\mathfrak{h}_e(u,v)=(\mathbf{H}_e^{1/2}u,\mathbf{H}_e^{1/2}v).$$

However, we can represent (see (5.2)) the forms $\mathfrak{h}_b$ and $\mathfrak{h}_e$ with the help of the operators $\mathbf{R}_b:\mathcal{Q}(\mathfrak{h}_b)\to\mathcal{H}_b$ and $\mathbf{R}_e:\mathcal{Q}(\mathfrak{h}_e)\to\mathcal{H}_e$. The only assumptions on the operators $\mathbf{R}_b$ (and $\mathbf{R}_e$) is that they have a closed range in the auxiliary Hilbert spaces $\mathcal{H}_b$ (and $\mathcal{H}_e$), cf. [16]. The representation theorem for the nonnegative definite forms implies

$$\mathfrak{h}_b(u,v)=(\mathbf{H}_b^{1/2}u,\mathbf{H}_b^{1/2}v)=(\mathbf{R}_bu,\mathbf{R}_bv)_{\mathcal{H}_b}, \tag{5.4}$$

$$\mathfrak{h}_e(u,v)=(\mathbf{H}_e^{1/2}u,\mathbf{H}_e^{1/2}v)=(\mathbf{R}_eu,\mathbf{R}_ev)_{\mathcal{H}_e}, \tag{5.5}$$

where $(\cdot,\cdot)_\mathcal{X}$ generically denotes the scalar product in the Hilbert space $\mathcal{X}$. The relations (5.4) and (5.5) imply that there exist isometric isomorphisms $Q_b:\mathcal{H}_b\to\mathcal{H}$ and $Q_e:$

$\mathcal{H}_e \to \mathcal{H}$ such that $\mathbf{H}_b^{1/2} = Q_b \mathbf{R}_b$, $\mathbf{H}_e^{1/2} = Q_e \mathbf{R}_e$, and in particular

$$(\mathbf{H}_b^{1/2}u, \mathbf{H}_b^{1/2}v) = (Q_b\mathbf{R}_b u, Q_b\mathbf{R}_b v) = (\mathbf{R}_b u, \mathbf{R}_b v)_{\mathcal{H}_b},$$
$$(\mathbf{H}_e^{1/2}u, \mathbf{H}_e^{1/2}v) = (Q_e\mathbf{R}_e u, Q_e\mathbf{R}_e v) = (\mathbf{R}_e u, \mathbf{R}_e v)_{\mathcal{H}_e}.$$

We also have for $\mathbf{u} \in \mathcal{Q}(\mathfrak{h}_b)$

$$Q_b^{-1}\mathbf{H}_b^{1/2}\mathbf{u} = \mathbf{R}_b\mathbf{u} = \sqrt{E\,I}\left(u_2' + \frac{u_1}{R}\right)',$$
$$Q_e^{-1}\mathbf{H}_e^{1/2}\mathbf{u} = \mathbf{R}_e\mathbf{u} = \sqrt{E\,A}\left(u_1' - \frac{u_2}{R}\right)$$

and $\mathbf{R}_b : \mathcal{Q}(\mathfrak{h}_b) \to \mathcal{H}_b = L^2[0,l]$ and $\mathbf{R}_e : \mathcal{Q}(\mathfrak{h}_e) \to \mathcal{H}_e = L^2[0,l]$.

Note that $\mathbf{H}_b$ is not positive definite but $\mathbf{H}_1$, which is defined by the form $\mathfrak{h}_1 = \mathfrak{h}_b + \mathfrak{h}_e$, is. For the details see [19, 32]. If we were to change the notation we would have to set $\widetilde{\mathfrak{h}}_b := \mathfrak{h}_1$. Since this would unnecessarily complicate the exposition we opt not to do so.

We show that

$$\|(\mathbf{H}_e^{1/2}\mathbf{H}_1^{-1/2})^\dagger\| \le \sqrt{\frac{I + A\,R^2}{A\,R^2}} \tag{5.6}$$

for our model problem. Set $\mathfrak{k} := \|(\mathbf{H}_e^{1/2}\mathbf{H}_1^{-1/2})^\dagger\|$ then

$$\|(\mathbf{H}_e^{1/2}\mathbf{H}_1^{-1/2})^* q_f\| = \sup_{\mathbf{v}\in\mathcal{Q}(\mathfrak{h}_b)} \frac{|(q_f, \mathbf{H}_e^{1/2}\mathbf{v})|}{\|\mathbf{H}_1^{1/2}\mathbf{v}\|} \ge \frac{1}{\mathfrak{k}}\,\|P_{\mathcal{Q}_\infty} q_f\|,$$

since

$$\mathsf{N}((\mathbf{H}_e^{1/2}\mathbf{H}_1^{-1/2})^*) = \mathsf{N}(\,\overline{\mathbf{H}_1^{-1/2}\mathbf{H}_e^{1/2}}\,) = \overline{\mathsf{N}(\mathbf{H}_e^{1/2})} = \overline{\mathcal{Q}_\infty}^{\,\|\cdot\|}.$$

For $Q_e^{-1}q_f \in L^2[0,l]$ we define $\mathbf{v}_0 = (\int_0^{(\cdot)}(Q_e^{-1}q_f)(s)ds, 0)$ (an element of $\mathcal{Q}(h_\kappa)$). For general $\mathbf{v}$ we have

$$\|\mathbf{H}_1^{1/2}\mathbf{v}\| = \Big(E\,I\int_0^l \Big(\big[v_2' + \frac{v_1}{R}\big]'\Big)^2 ds + E\,A\int_0^l \Big(v_1' - \frac{v_2}{R}\Big)^2 ds\Big)^{1/2}.$$

Now, set $\mathbf{v} = \mathbf{v}_0$ and compute

$$\|\mathbf{H}_1^{1/2}\mathbf{v}_0\| = \frac{\sqrt{E\,I + E\,A\,R^2}}{R}\|q_f\|.$$

This establishes

$$\sup_{\mathbf{v}\in\mathcal{Q}(\mathfrak{h}_b)} \frac{|(q_f, \mathbf{H}_e^{1/2}\mathbf{v})|}{\|\mathbf{H}_1^{1/2}\mathbf{v}\|} \ge \frac{|(q_f, \mathbf{H}_e^{1/2}\mathbf{v}_0)|}{\|\mathbf{H}_1^{1/2}\mathbf{v}_0\|} \ge \frac{|(Q_e^{-1}q_f, \mathbf{R}_e\mathbf{v}_0)_{L^2}|}{\frac{\sqrt{E\,I + E\,A\,R^2}}{R}\|q_f\|} = \sqrt{\frac{A\,R^2}{I + A\,R^2}}\|q_f\|,$$

which completes the proof of (5.6).

5.2. **Quantitative (and qualitative) conclusions.** The fact (5.6) allows us to apply Theorem 4.3 to obtain precise estimates for the behavior[6] of $\eta_i(\varepsilon, \lambda_q^\infty)$. Since $\mathbf{H}_1$ and not $\mathbf{H}_b$ is the positive definite operator, we will use the rod with the diameter ε_0 as a referent configuration. We chose

$$\varepsilon_0 = \frac{\sqrt{3}}{6}\sqrt{\frac{I + A\,R^2}{A\,R^2}}\,\frac{\lambda_{\mathsf{sec}}^\infty - \lambda_{\mathsf{min}}^\infty}{\lambda_{\mathsf{sec}}^\infty + \lambda_{\mathsf{min}}^\infty}, \tag{5.7}$$

where $\lambda_{\mathsf{sec}}^\infty$ and $\lambda_{\mathsf{min}}^\infty$ are the two lowermost eigenvalues of the Curved Rod model and $\lambda_{\mathsf{min}}^\varepsilon$ denotes the lowermost eigenvalue of the Arch Rod Model of the rod with diameter

[6] We have tacitly dropped the exponent from $\eta_i(\varepsilon^{-2}, \lambda_q^\infty)$ in order to simplify the notation.

ε. Theorems 3.11, 4.3, Remark 3.6 and Corollary 3.10—together with the observation that $\eta_i(\varepsilon, \lambda^\infty) < 1$ for any $\varepsilon > 0$—directly imply that

$$\frac{\lambda^\infty_{\mathsf{min}} - \lambda^\varepsilon_{\mathsf{min}}}{\lambda^\infty_{\mathsf{min}}} \leq \varepsilon^2 \frac{4(I + A\,R^2)}{A\,R^2} \frac{\lambda^\infty_{\mathsf{sec}} + \lambda^\infty_{\mathsf{min}}}{\lambda^\infty_{\mathsf{sec}} - \lambda^\infty_{\mathsf{min}}}, \qquad 0 < \varepsilon \leq \varepsilon_0.$$

Furthermore, if we chose $\varepsilon_1 = \frac{\sqrt{3}}{12}\sqrt{\frac{I+A\,R^2}{A\,R^2}\frac{\lambda^\infty_{\mathsf{sec}}-\lambda^\infty_{\mathsf{min}}}{\lambda^\infty_{\mathsf{sec}}+\lambda^\infty_{\mathsf{min}}}}$, then we obtain

$$\varepsilon^2 \frac{2(I + A\,R^2)\eta_i(\varepsilon_1, \lambda^\infty_{\mathsf{min}})}{A\,R^2} \leq \frac{\lambda^\infty_{\mathsf{min}} - \lambda^\varepsilon_{\mathsf{min}}}{\lambda^\infty_{\mathsf{min}}} \leq \varepsilon^2 \frac{4(I + A\,R^2)}{A\,R^2} \frac{\lambda^\infty_{\mathsf{sec}} + \lambda^\infty_{\mathsf{min}}}{\lambda^\infty_{\mathsf{sec}} - \lambda^\infty_{\mathsf{min}}}, \tag{5.8}$$

$0 < \varepsilon \leq \varepsilon_1$. If we are only interested in the upper estimate and we assume that there is $m \in \mathbb{N}$ such that $\lambda^\infty_m < \lambda^\infty_{m+1}$, then we have

$$\frac{\lambda^\infty_i - \lambda^\varepsilon_i}{\lambda^\infty_i} \leq \frac{3}{\max\limits_{i=1,\dots,m} \min\limits_{k \neq i} \dfrac{|\lambda^\infty_k - \lambda^\infty_i|}{\lambda^\infty_k + \lambda^\infty_i}} \frac{4(I + A\,R^2)}{A\,R^2} \varepsilon^2, \qquad i = 1, \dots, m. \tag{5.9}$$

This estimate holds for all $\varepsilon \leq \varepsilon_2$, where ε_2 is defined as the first ε for which the righthand side of (5.9) is smaller then 1. Estimate (5.9) can naturally be refined with the use of other $\eta_i(\varepsilon_2, \lambda^\infty_j)$ as is given by the framework of Theorem 3.3. The optimality of the estimate is meant in the sense of Theorem 3.9.

6. Conclusion

We have presented a constructive approach to spectral asymptotic estimates in the large coupling limit. Although we have concentrated on a use of this results for theoretical considerations from [5, 7, 9, 28], they are expected to be particularly useful in a design of computational procedures for various singularly perturbed spectral problems. This can be illustrated when comparing the numerical procedures for the Arch Model and the Curved Rod Model. It has been shown that the Curved Rod Model is is better behaved, with respect to the finite element approximations than the Arch Model, see [33]. Furthermore, a qualitative conclusion of Section 5.2 is that when interested in the transversal vibrations only, Arch Model can be ignored (up to the corrections of order ε^2). For more on the lower dimensional approximations in the theory of elasticity see [6, 19, 28, 32, 34].

In a practical computational setting it is not reasonable to assume that the spectral problem for $\mathbf{H}_\infty$ will be exactly solvable. We would like to emphasize that in the design of this theory we have not built the requirement of the explicit solvability of $\mathbf{H}_\infty$ into our results. To be more precise, nowhere in the proofs of Theorems 3.5 and 3.11 or Corollaries 3.7 and 3.8 is it necessary to have $\mathsf{R}(P) = \mathfrak{E}_\infty$. The only place where this assumption was necessary was to establish that (3.6) and (3.16) define the same approximation defects. Theorem 3.3 and similar results from [13, 17]—which are the workhorses of this theory—do not need this assumptions. Subsequently, the only limiting factor is the computability of $\eta_i(P)$ and the availability of information on the distance of $\mathrm{spec}(\Xi_\kappa)$—from Theorems 3.2 and 3.3— to the unwanted component of the spectrum. With this we hope to have illustrated the advantages and limitations of our theory

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